GALOIS ACTION ON KNOTS I — ABSOLUTE GALOIS ACTION

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Abstract. Our aim of this and subsequent papers is to enlighten (a part of, presumably) arithmetic structures of knots. This paper introduces a notion of profinite knots which extends usual knots and shows its various basic properties. Particularly an action of the absolute Galois group of the rational number filed on profinite knots is rigorously established by Drinfel'd's theory on the Grothendieck-Teichmüller group.

Contents

0. Introduction

It is known that there are analogies between algebraic number theory and 3-dimensional topology. It is said that Mazur and Manin among others spotted them in 1960's and, after a long silence, in 1990's, Kapranov and Reznikov ([Kap, R] and their lecture in MPI) who took up their ideas and explored them jointly and Morishita [Mo] whose works started independently in a more sophisticated aspect, settled the new area of mathematics, arithmetic topology. Lots of analogies between algebraic number theory and 3-dimensional topology are suggested in arithmetic topology, however, as far as we know, no direct relationship seems to be known. Our attempt of this and subsequent [F2] papers is to give a direct one particularly between Galois groups and knots.

A profinite tangle, a profinite analogue of an oriented tangle diagram, is introduced in Definition 2.2 as a consistent finite sequence of fundamental profinite tangles; symbols of three types A, $\widehat{B}$ and C (Definition 2.1). *A profinite knot*, a profinite analogue of an oriented knot diagram, is defined by a profinite tangle without endpoints and with a single connected component in Definition 2.3. A profinite version of Turaev moves is given in our Definition 2.7 (T1)-(T6), which determines an equivalence, called isotopy, for profinite tangles. The set of isotopy classes $\widehat{\mathcal{T}}$ of profinite tangles means the quotient of the set of profinite tangles by

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the equivalence and the set $\widehat{\mathcal{K}}$ of isotopy classes of profinite knots is the subspace of $\widehat{\mathcal{T}}$ consisting of isotopy classes of profinite knots (Definition 2.8). Our first theorem is

Theorem A (Theorem 2.10 and 2.14).

(1). The space $\widehat{\mathcal{K}}$ carries a structure of a topological commutative monoid whose product is given by the connected sum (2.1)*.*

(2). Let $\mathcal{K}$ denote the set of isotopy classes of (usual) oriented knots. Then there is a natural map

$$h : \mathcal{K} \to \widehat{\mathcal{K}}.$$

The map is with dense image and is a monoid homomorphism with respect to the connected sum.

The map h is naturally conjectured to be injective (Conjecture 2.11) since otherwise the Kontsevich invariant would fail to be a perfect knot invariant (cf. Remark 2.12 and 2.29).

The topological group $G\widehat{\mathcal{K}}$ of profinite knots is introduced as the group of fraction of the topological monoid $\widehat{\mathcal{K}}$ in Definition 2.30. A continuous action of the profinite Grothendieck-Teichmüller group $\widehat{GT}$ [Dr2] on $G\widehat{\mathcal{K}}$ is rigorously established in Definition 2.34 - Theorem 2.36. By using the inclusion from the absolute Galois group $G_{\mathbb{Q}}$ of the rational number field $\mathbb{Q}$ into $\widehat{GT}$, our second theorem is obtained.

Theorem B (Theorem 2.40).

Fix an embedding from the algebraic closure $\overline{\mathbb{Q}}$ of the rational number field $\mathbb{Q}$ into the complex number field $\mathbb{C}$. Then the group $G\widehat{\mathcal{K}}$ of profinite knots admits a non-trivial topological $G_{\mathbb{Q}}$-module structure. Namely, there is a non-trivial continuous Galois representation on profinite knots

$$\rho_0 : G_{\mathbb{Q}} \to Aut G\widehat{\mathcal{K}}.$$

The validity of a knot analogue of the so-called Belyĭ's theorem, i.e. the injectivity of ρ_0, is posed in Problem 2.43. Several projects and problems on the Galois action are posted in the end of this paper.

Our construction of the Galois action could be said as a profinite version of that of Kassel-Turaev [KT] done in the proalgebraic setting. Our Galois action on knots might also be related to the 'Galois relations' suggested in Gannon-Walton [GW]. Our discussion on profinite knots in this paper may be linked to Mazur's discussion on profinite equivalence of knots in [Ma]

The contents of the paper is as follows. §1 is devoted to a review of Drinfel'd's work on $\widehat{GT}$ and $G_{\mathbb{Q}}$-actions on profinite braids. Main results are presented in §2: The ABC-construction of profinite knots and their basic properties are introduced in §2.1. We also introduce and discuss the notion of pro-l knots in §2.2. It serves for our arguments in §2.3 where an absolute Galois action on profinite knots is established.

1. Profinite braids

This section is a review mainly on Drinfel'd's work [Dr2] of his profinite Grothendieck-Teichmüller group $\widehat{GT}$ and its action on profinite braids. Definitions of the profinite braid group $\widehat{B}_n$ and the absolute Galois group $G_{\mathbb{Q}}$ are recalled in Example 1.4. The definition of $\widehat{GT}$ is presented in Definition 1.6. In Theorem 1.9, it is explained that

$\widehat{GT}$ contains $G_{\mathbb{Q}}$. The action of $\widehat{GT}$ on $\widehat{B}_n$ is explained in Theorem 1.12. Specific properties of the action which will be required to next section are shown in Proposition 1.16 and Proposition 1.18.

Definition 1.1. The *Artin braid group B_n with n-strings* ($n \geqslant 2$) is the group generated by σ_i ($1 \leqslant i \leqslant n-1$) with two relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},$$
$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{if} \quad |i-j| > 1.$$

The unit of B_n is denoted by e_n. For $n = 1$, put $B_1 = \{e_1\}$: the trivial group. The *pure braid group P_n with n-strings* is the kernel of the natural projection from B_n to the symmetric group $\mathfrak{S}_n$ of degree n.

When $n = 2$, there is an identification $B_2 \simeq \mathbb{Z}$ and the subgroup P_2 corresponds to $2\mathbb{Z}$ under the identification. When $n = 3$, P_3 contains a free group F_2 generated by σ_1^2 and σ_2^2.

Notation 1.2. The generator σ_i in B_n is depicted as in Figure 1.1. And for b and $b' \in B_n$, we draw the product $b \cdot b' \in B_n$ as in Figure 1.2 (the order of product $b \cdot b'$ is chosen to combine the bottom endpoints of b with the top endpoints of b').

FIGURE 1.1. σ_i

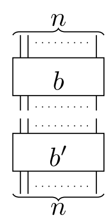


FIGURE 1.2. $b \cdot b'$

For $1 \leqslant i < j \leqslant n$, special elements

$$x_{i,j} = x_{j,i} = (\sigma_{j-1} \cdots \sigma_{i+1})\sigma_i^2(\sigma_{j-1} \cdots \sigma_{i+1})^{-1}$$

form a generator of P_n. For $1 \leqslant a \leqslant a + \alpha < b \leqslant b + \beta \leqslant n$, we define

$$x_{a\cdots a+\alpha, b\cdots b+\beta} := (x_{a,b}x_{a,b+1} \cdots x_{a,b+\beta}) \cdot (x_{a+1,b}x_{a+1,b+1} \cdots x_{a+1,b+\beta}) \cdots (x_{a+\alpha,b}x_{a+\alpha,b+1} \cdots x_{a+\alpha,b+\beta}) \in P_n.$$

They are drawn in Figure 1.3 and 1.4.

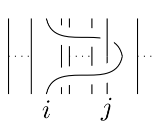


FIGURE 1.3. x_{ij}

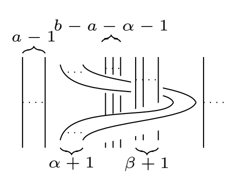


FIGURE 1.4. $x_{a\cdots a+\alpha, b\cdots b+\beta}$

Next we briefly review the definition and a few examples of profinite groups.

Definition 1.3. A topological group G is called a *profinite group* if it is a projective limit $\varprojlim G_i$ of a projective system of finite groups $\{G_i\}_{i \in I}$. For a discrete group Γ, its *profinite completion* $\widehat{\Gamma}$ is the profinite group defined by the projective limit

$$\widehat{\Gamma} = \varprojlim \Gamma/N$$

where N runs over all normal subgroups of Γ with finite indices.

For profinite groups, consult [RZ] for example. We note there is a natural homomorphism $\Gamma \to \widehat{\Gamma}$. In the paper, we employ the same symbol when we express the image of elements of Γ by the map if there is no confusion.

Example 1.4. (1). The set $\widehat{\mathbb{Z}}$ of *profinite integers* is the profinite completion of $\mathbb{Z}$. There is an identification $\widehat{\mathbb{Z}} \simeq \prod_p \mathbb{Z}_p$. Here p runs over all primes and $\mathbb{Z}_p$ stands for the ring of p-adic integers.

(2). The *absolute Galois group* $G_{\mathbb{Q}}$ of the rational number field $\mathbb{Q}$ is the profinite group

$$G_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q}) := \varprojlim Gal(F/\mathbb{Q}).$$

Here the limit runs over all finite Galois extension F of $\mathbb{Q}$ and $Gal(F/\mathbb{Q})$ means its Galois group.

(3). The *profinite braid group* $\widehat{B}_n$ means the profinite completion of B_n. It contains the *profinite pure braid group* $\widehat{P}_n$ (the profinite completion of P_n), which is equal to the kernel of the natural projection $\widehat{B}_n \to \mathfrak{S}_n$. It is known that both B_n and P_n are *residually finite*, i.e, their natural maps are both injective;

$$B_n \hookrightarrow \widehat{B}_n \quad \text{and} \quad P_n \hookrightarrow \widehat{P}_n. \tag{1.1}$$

When $n = 2$, we have $\widehat{B}_2 \simeq \widehat{\mathbb{Z}}$. For profinite braid groups, we employ the same figures as in Notation 1.2 to express such elements. Further we also depict $\sigma_i^c \in \widehat{B}_n$ $(c \in \widehat{\mathbb{Z}})$ as Figure 1.5.

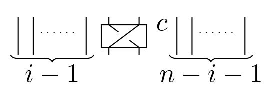


FIGURE 1.5. $\sigma_i^c \in \widehat{B}_n$ $(c \in \widehat{\mathbb{Z}})$

To state the definition of $\widehat{GT}$, we need fix several notations.

Notation 1.5. Let F_2 be the free group of rank 2 with two variables x and y and $\widehat{F}_2$ be its profinite completion. For any $f \in \widehat{F}_2$ and any group homomorphism $\tau : \widehat{F}_2 \to G$ sending $x \mapsto \alpha$ and $y \mapsto \beta$, the symbol $f(\alpha, \beta)$ stands for the image $\tau(f)$. Particularly for the (actually injective) group homomorphism $\widehat{F}_2 \to \widehat{P}_n$ sending $x \mapsto x_{a\cdots a+\alpha, b\cdots b+\beta}$ and $y \mapsto x_{b\cdots b+\beta, c\cdots c+\gamma}$ $(1 \leqslant a \leqslant a+\alpha < b \leqslant b+\beta < c \leqslant c+\gamma \leqslant n)$, the image of $f \in \widehat{F}_2$ is denoted by $f_{a\cdots a+\alpha, b\cdots b+\beta, c\cdots c+\gamma}$.

The profinite Grothendieck-Teichmüller group $\widehat{GT}$ which is a main character of our paper is defined by Drinfel'd [Dr2] to be a profinite subgroup of the topological automorphism group of $\widehat{F}_2$.

Definition 1.6 ([Dr2])**.** The *profinite Grothendieck-Teichmüller group* [1] $\widehat{GT}$ is the profinite subgroup of $Aut\widehat{F}_2$ defined by

$$\widehat{GT} := \left\{ \sigma \in Aut\widehat{F}_2 \;\middle|\; \begin{array}{l} \sigma(x) = x^\lambda, \sigma(y) = f^{-1}y^\lambda f \text{ for some } (\lambda, f) \in \widehat{\mathbb{Z}}^\times \times \widehat{F}_2 \\ \text{satisfying the three relations below.} \end{array} \right\}$$

$$f(x,y)f(y,x) = 1 \quad \text{in } \widehat{F}_2, \tag{1.2}$$

$$f(z,x)z^m f(y,z)y^m f(x,y)x^m = 1 \quad \text{in } \widehat{F}_2 \text{ with } z = (xy)^{-1} \text{ and } m = \frac{\lambda - 1}{2}, \tag{1.3}$$

$$f_{1,2,34}f_{12,3,4} = f_{2,3,4}f_{1,23,4}f_{1,2,3} \quad \text{in } \widehat{P}_4. \tag{1.4}$$

Remark 1.7. (1). In some literatures, (1.2), (1.3) and (1.4) are called *2-cycle, 3-cycle and 5-cycle relation* respectively. The author often calls (1.2) and (1.3) by *two hexagon equations* and (1.4) by *one pentagon equation* because they reflect the three axioms, two hexagon and one pentagon axioms, of braided monoidal (tensor) categories [JS]. We remind that (1.4) represents

$$f(x_{12}, x_{23}x_{24})f(x_{13}x_{23}, x_{34}) = f(x_{23}, x_{34})f(x_{12}x_{13}, x_{24}x_{34})f(x_{12}, x_{23}) \quad \text{in } \widehat{P}_4.$$

In several literatures such as [I1, S], the equation (1.4) is replaced by a different (more symmetric) formulation.

(2). For the *proalgebraic* Grothendieck-Teichmüller group $GT(\mathbf{k})$ in [Dr2] ($\mathbf{k}$: a field of characteristic 0), it is shown that its one pentagon equation implies its two hexagon equations in [F1]. But as for our profinite group $\widehat{GT}$, it is not known if the pentagon equation (1.4) implies two hexagon equations (1.2) and (1.3).

(3). We remark that each $\sigma \in \widehat{GT}$ determines a pair (λ, f) uniquely because the pentagon equation (1.4) implies that f belongs to the topological commutator of $\widehat{F}_2$. By abuse of notation, we occasionally express the pair (λ, f) to represents σ and denote as $\sigma = (\lambda, f) \in \widehat{GT}$. The above set-theoretically defined $\widehat{GT}$ forms indeed a profinite group whose product is induced from that of $Aut\widehat{F}_2$ and is given by

$$(\lambda_2, f_2) \circ (\lambda_1, f_1) = \Big(\lambda_2\lambda_1, f_2 \cdot f_1(x^{\lambda_2}, f_2^{-1}y^{\lambda_2}f_2)\Big). \tag{1.5}$$

The next lemma will be used later.

Lemma 1.8. *Let $p_i : \widehat{P}_3 \to \widehat{P}_2 (= \widehat{\mathbb{Z}})$ $(i = 1, 2, 3)$ denote the map of omission of i-th strand in $\widehat{P}_3$. Let $(\lambda, f) \in \widehat{GT}$. Then $p_i(f_{1,2,3}) = 0$ $(i = 1, 2, 3)$.*

Proof. It is easy because f belongs to the commutator of $\widehat{F}_2$ as mentioned above. □

One of the important property of the profinite Grothendieck-Teichmüller group $\widehat{GT}$ is that it contains the absolute Galois group $G_{\mathbb{Q}}$.

Theorem 1.9 ([Dr2, I2])**.** *Fix an embedding from $\overline{\mathbb{Q}}$ into $\mathbb{C}$. Then there is an embedding*

$$G_{\mathbb{Q}} \hookrightarrow \widehat{GT}. \tag{1.6}$$

[1] It is named after Grothendieck's project of *un jeu de Teichmüller-Lego* posted in *Esquisse d'un programme* [G2]. A construction of such group was suggested there though Drinfel'd came to the group independently in his subsequent works [Dr1, Dr2] on deformation of specific type of quantum groups.

We briefly review its proof below.

Proof. As is explained in [I2], an absolute Galois action on $\widehat{F}_2$, i.e.

$$G_{\mathbb{Q}} \to Aut\ \widehat{F}_2 \tag{1.7}$$

is derived from the action on the profinite (scheme-theoretical, cf. [G1]) fundamental group $\widehat{\pi}_1(\mathbf{P}^1_{\mathbb{Q}}\backslash\{0,1,\infty\},\overrightarrow{01})$ of the algebraic curve $\mathbf{P}^1_{\mathbb{Q}}\backslash\{0,1,\infty\}$ with Deligne's [De] tangential base point $\overrightarrow{01}$ (this is achieved in the method explained in Remark 1.14 below). The so-called *Belyĭ's theorem* [Be] claims that the map (1.7) is injective:

$$G_{\mathbb{Q}} \hookrightarrow Aut\ \widehat{F}_2. \tag{1.8}$$

The equations (1.2)-(1.4) are checked for $\sigma \in G_{\mathbb{Q}}$ in [Dr2, IM], which means that $G_{\mathbb{Q}}$ is contained in $\widehat{GT} \subset Aut\widehat{F}_2$. □

Example 1.10. Fix an embedding from $\overline{\mathbb{Q}}$ into $\mathbb{C}$. Then the complex conjugation sending $z \in \mathbb{C}$ to $\bar{z} \in \mathbb{C}$ determines an element $\varsigma_0 \in G_{\mathbb{Q}}$. It is mapped to the pair $(-1,1) \in \widehat{GT}$ by (1.6).

Asking its surjectivity on the injection (1.6) is open for many years:

Problem 1.11. Is $G_{\mathbb{Q}}$ equal to $\widehat{GT}$?

The following Drinfel'd's [Dr2] $\widehat{GT}$-action on $\widehat{B}_n$ (a detailed description is also given in [IM]) plays a fundamental role of our results, $\widehat{GT}$-action on knots.

Theorem 1.12 ([Dr2, IM])**.** *Let $n \geqslant 2$. There is a continuous $\widehat{GT}$-action on $\widehat{B}_n$*

$$\rho_n : \widehat{GT} \to Aut\widehat{B}_n$$

given by

$$\sigma = (\lambda, f) : \begin{cases} \sigma_1 & \mapsto & & \sigma_1^{\lambda}, \\ \sigma_i & \mapsto & f^{-1}_{1\cdots i-1,i,i+1} & \sigma_i^{\lambda} f_{1\cdots i-1,i,i+1} \qquad (2 \leqslant i \leqslant n-1). \end{cases}$$

We denote $\rho_n(\sigma)(b)$ simply by $\sigma(b)$ when there is no confusion.

Remark 1.13. (1). According to the method to calculate the action in [Dr2] (explicitly presented in the appendix of [IM]), particularly we have

$$\begin{aligned} (\sigma_1\cdots\sigma_i) &\mapsto f^{-1}_{[1],[i],[n-i-1]}\cdot(\sigma_1\cdots\sigma_i)\cdot x^m_{1\cdots i,i+1}, \\ (\sigma_i\cdots\sigma_1) &\mapsto x^m_{1\cdots i,i+1}\cdot(\sigma_i\cdots\sigma_1)\cdot f_{[1],[i],[n-i-1]}. \end{aligned} \tag{1.9}$$

Here $m = \frac{\lambda-1}{2}$ and for $f_{[1],[i],[n-i-1]}$, see (1.13).
(2). We note that ρ_n is injective when $n \geqslant 4$.

By Theorem 1.9 and Theorem 1.12, we obtain the absolute Galois representation

$$\rho_n : G_{\mathbb{Q}} \to Aut\widehat{B}_n. \tag{1.10}$$

The below is an algebraic-geometrical interpretation of the Galois action in terms of Grothendieck's theory [G1] on profinite (scheme-theoretical) fundamental groups.

Remark 1.14. We have a well-known identification between the braid group B_n with the topological fundamental group $\pi_1(X_n(\mathbb{C}),*)$. Here $X_n(\mathbb{C}) = Conf^n_{\mathfrak{S}_n}(\mathbb{C})$ means the quotient of the *configuration space*

$$Conf^n(\mathbb{C}) = \{(z_1,\dots,z_n)\in\mathbb{C}^n | z_i\neq z_j (i\neq j)\}$$

by the symmetric group $\mathfrak{S}_n$ action and $*$ is a basepoint.

Let $\widehat{\pi}_1(X_n\times\overline{\mathbb{Q}},*)$ denote the profinite (scheme-theoretical) fundamental group of $X_n\times\overline{\mathbb{Q}}$ in the sense of Grothendieck [G1]. Here the scheme X_n means the $\mathbb{Q}$-structure of $X_n(\mathbb{C})$ and $*$ is a basepoint defined over $\overline{\mathbb{Q}}$ in the sense of loc.cit. Fix an embedding from $\overline{\mathbb{Q}}$ into $\mathbb{C}$, then, by the so-called Riemann's existence theorem (lo.cit. VII.Théorème 5.1), the group $\widehat{\pi}_1(X_n\times\overline{\mathbb{Q}},*)$ is identified with the profinite completion of $\pi_1(X_n(\mathbb{C}),*)$. Hence we have an identification

$$\widehat{B}_n \simeq \widehat{\pi}_1(X_n\times\overline{\mathbb{Q}},*). \tag{1.11}$$

Next assume that $*$ is defined over $\mathbb{Q}$. Then by [G1] IX.Théorème 6.1, we have the homotopy exact sequence of the profinite fundamental group

$$1\to\widehat{\pi}_1(X_n\times\overline{\mathbb{Q}},*)\to\widehat{\pi}_1(X_n,*)\to\widehat{\pi}_1(Spec\mathbb{Q},*)\to 1.$$

The last $\widehat{\pi}_1(Spec\ \mathbb{Q},*)$ is nothing but the absolute Galois group $G_{\mathbb{Q}}$. A point here is that each basepoint $*$ determines a section s_* of the exact sequence. By (1.11), the section s_* yields a continuous Galois representation on $\widehat{B}_n$

$$\rho_{n,*}: G_{\mathbb{Q}}\to Aut\widehat{B}_n$$

by inner conjugation, i.e., $\rho_{n,*}(\sigma)(b) = s_*(\sigma)\cdot b\cdot s_*(\sigma)^{-1}$ ($\sigma\in G_{\mathbb{Q}}$ and $b\in\widehat{B}_n$). A specific (tangential in the sense of Deligne [De]) basepoint t_n is constructed in [IM], where they showed that the resulting ρ_{n,t_n} is equal to our ρ_n in (1.10).

Special properties of the $\widehat{GT}$-action in Theorem 1.12 are presented in the following Proposition 1.16 and Proposition 1.18 (though they are implicitly suggested in [Dr2]). They will be employed several times in our paper.

Notation 1.15. Put $n>0$ and $m_1,m_2\geqslant 0$. On the continuous homomorphism

$$e_{m_1}\otimes\cdot\otimes e_{m_2}:\widehat{B}_n\to\widehat{B}_{m_1+n+m_2}$$

which is defined by $\sigma_i\mapsto\sigma_{m_1+i}$ (obtained by placing the trivial braids e_{m_1} and e_{m_2} on the left and right respectively), we denote the image of $b\in\widehat{B}_n$ by $e_{m_1}\otimes b\otimes e_{m_2}$.

Proposition 1.16. *Put $n>0$ and $m_1,m_2\geqslant 0$. Let $\sigma=(\lambda,f)\in\widehat{GT}$ and $b\in\widehat{B}_n$. Then*

$$\sigma(e_{m_1}\otimes b\otimes e_{m_2}) = f^{-1}_{[m_1],[n],[m_2]}\cdot(e_{m_1}\otimes\sigma(b)\otimes e_{m_2})\cdot f_{[m_1],[n],[m_2]}. \tag{1.12}$$

Here

$$f_{[m_1],[n],[m_2]} := f_{1\cdots m_1,m_1+1\cdots m_1+n-1,m_1+n}\cdot \tag{1.13}$$
$$f_{1\cdots m_1,m_1+1\cdots m_1+n-2,m_1+n-1}\cdots f_{1\cdots m_1,m_1+1,m_1+2}\in\widehat{B}_{m_1+n+m_2}.$$

Proof. It is enough to check (1.12) for $b=\sigma_i$ ($1\leqslant i\leqslant n-1$). By Theorem 1.12,

(1.14)

$$f^{-1}_{[m_1],[n],[m_2]}\cdot(e_{m_1}\otimes\sigma(\sigma_i)\otimes e_{m_2})\cdot f_{[m_1],[n],[m_2]} =$$
$$f^{-1}_{[m_1],[n],[m_2]}\cdot f^{-1}_{m_1+1\cdots m_1+i-1,m_1+i,m_1+i+1}\cdot\sigma^{\lambda}_{m_1+i}\cdot f_{m_1+1\cdots m_1+i-1,m_1+i,m_1+i+1}\cdot f_{[m_1],[n],[m_2]}.$$

- When $M \geqslant i+2$, both $f_{m_1+1\cdots m_1+i-1,m_1+i,m_1+i+1}$ and σ_{m_1+i} commute with $f_{1\cdots m_1,m_1+1\cdots m_1+M-1,m_1+M}$ because $x_{m_1+1\cdots m_1+i-1,m_1+i}$, x_{m_1+i,m_1+i+1} and σ_{m_1+i} commute with $x_{1\cdots m_1,m_1+1\cdots m_1+M-1}$ and $x_{m_1+1\cdots m_1+M-1,m_1+M}$. Therefore

$$(1.14) = f^{-1}_{[m_1],[i+1],[m_2]} \cdot f^{-1}_{m_1+1\cdots m_1+i-1,m_1+i,m_1+i+1} \cdot \sigma^{\lambda}_{m_1+i} \cdot f_{m_1+1\cdots m_1+i-1,m_1+i,m_1+i+1} \cdot f_{[m_1],[i+1],[m_2]}.$$

- When $M = i, i+1$, our calculation goes as follows.

$$(1.14) = f^{-1}_{[m_1],[i-1],[m_2]} \cdot f^{-1}_{1\cdots m_1,m_1+1\cdots m_1+i-1,m_1+i} \cdot f^{-1}_{1\cdots m_1,m_1+1\cdots m_1+i,m_1+i+1} \cdot f^{-1}_{m_1+1\cdots m_1+i-1,m_1+i,m_1+i+1} \cdot \sigma^{\lambda}_{m_1+i} \cdot f_{m_1+1\cdots m_1+i-1,m_1+i,m_1+i+1} \cdot f_{1\cdots m_1,m_1+1\cdots m_1+i,m_1+i+1} \cdot f_{1\cdots m_1,m_1+1\cdots m_1+i-1,m_1+i} \cdot f_{[m_1],[i-1],[m_2]},$$

by the pentagon equation (1.4),

$$= f^{-1}_{[m_1],[i-1],[l+m_2]} \cdot f^{-1}_{1\cdots m_1+i-1,m_1+i,m_1+i+1} \cdot f^{-1}_{1\cdots m_1,m_1+1\cdots m_1+i-1,m_1+i\ m_1+i+1} \cdot \sigma^{\lambda}_{m_1+i} \cdot f_{1\cdots m_1,m_1+1\cdots m_1+i-1,m_1+i\ m_1+i+1} \cdot f_{1\cdots m_1+i-1,m_1+i,m_1+i+1} \cdot f_{[m_1],[i-1],[l+m_2+2]}.$$

Since σ_{m_1+i} commutes with $f_{1\cdots m_1,m_1+1\cdots m_1+i-1,m_1+i\ m_1+i+1}$,

$$= f^{-1}_{[m_1],[i-1],[l+m_2]} \cdot f^{-1}_{1\cdots m_1+i-1,m_1+i,m_1+i+1} \cdot \sigma^{\lambda}_{m_1+i} \cdot f_{1\cdots m_1+i-1,m_1+i,m_1+i+1} \cdot f_{[m_1],[i-1],[l+m_2+2]}.$$

- When $M \leqslant i-1$, both $f_{1\cdots m_1+i-1,m_1+i,m_1+i+1}$ and σ_{m_1+i} commute with $f_{1\cdots m_1,m_1+1\cdots m_1+M-1,m_1+M}$. Therefore

$$\begin{aligned}(1.14) &= f^{-1}_{1\cdots m_1+i-1,m_1+i,m_1+i+1} \cdot \sigma^{\lambda}_{m_1+i} \cdot f_{1\cdots m_1+i-1,m_1+i,m_1+i+1} \\ &= \sigma(\sigma_{m_1+i}) = \sigma(e_{m_1} \otimes \sigma_i \otimes e_{m_2}).\end{aligned}$$

Hence we get the equality (1.12). $\square$

Notation 1.17. Let $l, n \geqslant 1$ and $1 \leqslant k \leqslant l$. We consider the continuous group homomorphism

$$P_l \to P_{l+n-1} \tag{1.15}$$

sending, for $1 \leqslant i < j \leqslant l$,

$$x_{i,j} \mapsto \begin{cases} x_{i+k-1,j+k-1} & (k < i), \\ x_{i\cdots i+k-1,j+k-1} & (k = i), \\ x_{i,j+k-1} & (i < k < j), \\ x_{i,j\cdots j+k-1} & (k = j), \\ x_{i,j} & (j < k). \end{cases}$$

We obtain the map by replacing the k-th string (from the left) by the trivial braid e_n with n strings, hence it naturally extends to two maps (not homomorphisms)

$$ev_{k,e_n} : B_l \to B_{l+n-1} \qquad \text{and} \qquad ev^{k,e_n} : B_l \to B_{l+n-1}$$

which replaces the k-th string (from the bottom and the above left respectively) by the trivial braid e_n with n strings. Both of their restrictions into P_l are equal to the above map (1.15). Since the map (1.15) also continuously extends into

the homomorphism $\widehat{P}_l \to \widehat{P}_l$, our two maps naturally extend to the maps (not homomorphisms)

$$ev_{k,e_n} : \widehat{B}_l \to \widehat{B}_{l+n-1} \qquad \text{and} \qquad ev^{k,e_n} : \widehat{B}_l \to \widehat{B}_{l+n-1}.$$

Here we employ the same symbols because there would be no confusion.

Proposition 1.18. *Put $l \geqslant 1$. Let $\sigma = (\lambda, f) \in \widehat{GT}$ and $b \in \widehat{B}_l$. Set $k' = b(k)$. Here $b(k)$ stands for the image of k by the permutation corresponding to b by the projection $B_l \to \mathfrak{S}_l$. Then, for each k with $1 \leqslant k \leqslant l$, we have*

$$(1.16) \qquad \begin{aligned} \sigma(ev_{k,e_n}(b)) &= f^{-1}_{[k'-1],[n],[l-k']} \cdot ev_{k,e_n}(\sigma(b)) \cdot f_{[k-1],[n],[l-k]}, \\ \sigma(ev^{k',e_n}(b)) &= f^{-1}_{[k'-1],[n],[l-k']} \cdot ev^{k',e_n}(\sigma(b)) \cdot f_{[k-1],[n],[l-k]}. \end{aligned}$$

Proof. It suffices to prove the first equality, for the validity of the second equality is immediate once we have the first equality. Firstly we prove (1.16) for $b = \sigma_i$ $(1 \leqslant i \leqslant l-1)$.

- When $k < i$, we have $ev_{k,e_n}(\sigma_i) = \sigma_{i+n-1}$ and $k' = k$. Therefore

$$\begin{aligned} RHS &= f^{-1}_{[k-1],[n],[l-k]} \cdot ev_{k,e_n}(f^{-1}_{1\cdots i-1,i,i+1}\sigma_i^{\lambda} f_{1\cdots i-1,i,i+1}) \cdot f_{[k-1],[n],[l-k]} \\ &= f^{-1}_{[k-1],[n],[l-k]} \cdot (f^{-1}_{1\cdots i+n-2,i+n-1,i+n}\sigma_{i+n-1}^{\lambda} f_{1\cdots i+n-2,i+n-1,i+n}) \cdot f_{[k-1],[n],[l-k]}. \end{aligned}$$

  By $k-1+n \leqslant i+n-2$, $f_{[k-1],[n],[l-k]}$ commutes with $f_{1\cdots i+n-2,i+n-1,i+n}$ and σ_{i+n-1}. Thus

$$\begin{aligned} &= f^{-1}_{1\cdots i+n-2,i+n-1,i+n}\sigma_{i+n-1}^{\lambda} f_{1\cdots i+n-2,i+n-1,i+n} = \sigma(\sigma_{i+n-1}) \\ &= \sigma(ev_{k,e_n}(\sigma_i)) = LHS. \end{aligned}$$

- When $k = i$, we have $ev_{k,e_n}(\sigma_i) = ev_{i,e_n}(\sigma_i) = \sigma_i\sigma_{i+1}\cdots\sigma_{i+n-1}$ and $k' = k+1 = i+1$. Therefore

$$\begin{aligned} RHS &= f^{-1}_{[i],[n],[l-i-1]} \cdot ev_{i,e_n}(f^{-1}_{1\cdots i-1,i,i+1}\sigma_i^{\lambda} f_{1\cdots i-1,i,i+1}) \cdot f_{[i-1],[n],[l-i]} \\ &= f^{-1}_{[i],[n],[l-i-1]} \cdot f^{-1}_{1\cdots i-1,i,i+1\cdots i+n} \cdot ev_{i,e_n}(\sigma_i^{\lambda}) \cdot f_{1\cdots i-1,i\cdots i+n-1,i+n} \cdot f_{[i-1],[n],[l-i]} \\ &= f^{-1}_{[i],[n],[l-i-1]} \cdot f^{-1}_{1\cdots i-1,i,i+1\cdots i+n} \cdot ev_{i,e_n}(\sigma_i \cdot x^m_{i,i+1}) \cdot f_{[i-1],[n+1],[l-i-1]} \\ &= f^{-1}_{[i],[n],[l-i-1]} \cdot f^{-1}_{1\cdots i-1,i,i+1\cdots i+n} \cdot (\sigma_i\cdots\sigma_{i+n-1}) \cdot x^m_{i\cdots i+n-1,i+n} \\ &\qquad \cdot f_{[i-1],[n+1],[l-i-1]} \\ &= f^{-1}_{[i],[n],[l-i-1]} \cdot f^{-1}_{1\cdots i-1,i,i+1\cdots i+n} \cdot (e_{i-1} \otimes (\sigma_1\cdots\sigma_n) \cdot x^m_{1\cdots n,n+1} \otimes e_{l-i-1}) \\ &\qquad \cdot f_{[i-1],[n+1],[l-i-1]}. \end{aligned}$$

  By (1.9),

$$\begin{aligned} &= f^{-1}_{[i],[n],[l-i-1]} \cdot f^{-1}_{1\cdots i-1,i,i+1\cdots i+n} \cdot (e_{i-1} \otimes f_{[1],[n],[0]} \cdot \sigma(\sigma_1\cdots\sigma_n) \otimes e_{l-i-1}) \\ &\qquad \cdot f_{[i-1],[n+1],[l-i-1]} \\ &= f^{-1}_{[i],[n],[l-i-1]} \cdot f^{-1}_{1\cdots i-1,i,i+1\cdots i+n} \cdot (e_{i-1} \otimes f_{[1],[n],[0]} \cdot \otimes e_{l-i-1}) \\ &\qquad \cdot (e_{i-1} \otimes \sigma(\sigma_1\cdots\sigma_n) \otimes e_{l-i-1}) \cdot f_{[i-1],[n+1],[l-i-1]}. \end{aligned}$$

  By Lemma 1.19 below,

$$= f^{-1}_{[i-1],[n+1],[l-i-1]} \cdot (e_{i-1} \otimes \sigma(\sigma_1\cdots\sigma_n) \otimes e_{l-i-1}) \cdot f_{[i-1],[n+1],[l-i-1]}.$$

By Proposition 1.16,

$$= \sigma(e_{i-1} \otimes (\sigma_1 \cdots \sigma_n) \otimes e_{l-i-1}) = \sigma(\sigma_i \cdots \sigma_{i+n-1}) = \sigma(ev_{i,e_n}(\sigma_i)) = LHS.$$

- When $k = i+1$, we have $ev_{k,e_n}(\sigma_i) = ev_{i+1,e_n}(\sigma_i) = \sigma_{i+n-1} \cdots \sigma_{i+1}\sigma_i$ and $k' = k-1 = i$. Therefore

$$\begin{aligned}
RHS &= f^{-1}_{[i-1],[n],[l-i]} \cdot ev_{i+1,e_n}(f^{-1}_{1\cdots i-1,i,i+1}\sigma_i^\lambda f_{1\cdots i-1,i,i+1}) \cdot f_{[i],[n],[l-i-1]} \\
&= f^{-1}_{[i-1],[n],[l-i]} \cdot f^{-1}_{1\cdots i-1,i\cdots i+n-1,i+n} \cdot ev_{i+1,e_n}(\sigma_i^\lambda) \cdot f_{1\cdots i-1,i,i+1\cdots i+n} \cdot f_{[i],[n],[l-i-1]} \\
&= f^{-1}_{[i-1],[n+1],[l-i-1]} \cdot ev_{i+1,e_n}(x^m_{i,i+1} \cdot \sigma_i) \cdot f_{1\cdots i-1,i,i+1\cdots i+n} \cdot f_{[i],[n],[l-i-1]} \\
&= f^{-1}_{[i-1],[n+1],[l-i-1]} \cdot x^m_{i\cdots i+n-1,i+n} \cdot (\sigma_{i+n-1} \cdots \sigma_{i+1}\sigma_i) \\
&\qquad \cdot f_{1\cdots i-1,i,i+1\cdots i+n} \cdot f_{[i],[n],[l-i-1]} \\
&= f^{-1}_{[i-1],[n+1],[l-i-1]} \cdot (e_{i-1} \otimes x^m_{1\cdots n,n+1} \cdot (\sigma_n \cdots \sigma_1) \otimes e_{l-i-1}) \\
&\qquad \cdot f_{1\cdots i-1,i,i+1\cdots i+n} \cdot f_{[i],[n],[l-i-1]}.
\end{aligned}$$

By (1.9),

$$\begin{aligned}
&= f^{-1}_{[i-1],[n+1],[l-i-1]} \cdot (e_{i-1} \otimes \sigma(\sigma_n \cdots \sigma_1) \cdot f^{-1}_{[1],[n],[0]} \otimes e_{l-i-1}) \\
&\qquad \cdot f_{1\cdots i-1,i,i+1\cdots i+n} \cdot f_{[i],[n],[l-i-1]} \\
&= f^{-1}_{[i-1],[n+1],[l-i-1]} \cdot (e_{i-1} \otimes \sigma(\sigma_n \cdots \sigma_1) \otimes e_{l-i-1}) \cdot (e_{i-1} \otimes f^{-1}_{[1],[n],[0]} \cdot \otimes e_{l-i-1}) \\
&\qquad \cdot f_{1\cdots i-1,i,i+1\cdots i+n} \cdot f_{[i],[n],[l-i-1]}.
\end{aligned}$$

By Lemma 1.19 below,

$$= f^{-1}_{[i-1],[n+1],[l-i-1]} \cdot (e_{i-1} \otimes \sigma(\sigma_n \cdots \sigma_1) \otimes e_{l-i-1}) \cdot f_{[i-1],[n+1],[l-i-1]}.$$

By Proposition 1.16,

$$= \sigma(e_{i-1} \otimes (\sigma_n \cdots \sigma_1) \otimes e_{l-i-1}) = \sigma(\sigma_{i+n-1} \cdots \sigma_i) = \sigma(ev_{i+1,e_n}(\sigma_i)) = LHS.$$

- When $k > i+1$, we have $ev_{k,e_n}(\sigma_i) = \sigma_i$ and $k' = k$. Therefore

$$\begin{aligned}
RHS &= f^{-1}_{[k-1],[n],[l-k]} \cdot ev_{k,e_n}(f^{-1}_{1\cdots i-1,i,i+1}\sigma_i^\lambda f_{1\cdots i-1,i,i+1}) \cdot f_{[k-1],[n],[l-k]} \\
&= f^{-1}_{[k-1],[n],[l-k]} \cdot (f^{-1}_{1\cdots i-1,i,i+1}\sigma_i^\lambda f_{1\cdots i-1,i,i+1}) \cdot f_{[k-1],[n],[l-k]}.
\end{aligned}$$

By $i+1 \leqslant k-1$, $f_{[k-1],[n],[l-k]}$ commutes with $f_{1\cdots i-1,i,i+1}$ and σ_i. Thus

$$= f^{-1}_{1\cdots i-1,i,i+1}\sigma_i^\lambda f_{1\cdots i-1,i,i+1} = \sigma(\sigma_i) = \sigma(ev_{k,e_n}(\sigma_i)) = LHS.$$

Whence the equation (1.16) for $b = \sigma_i$ is obtained.

The validity for $b = \sigma_i$ implies the validity for $b \in B_l$ because each element of B_l is a finite product of σ_i's. Whence particularly we have the validity for P_l. Then by continuity we have for $\widehat{P}_l$. Since we have the validity for B_l and $\widehat{P}_l$, we have the validity for $\widehat{B}_l$. □

The auxiliary lemma below is required to prove the above proposition.

Lemma 1.19. *For $\sigma = (\lambda, f) \in \widehat{GT}$ and $i,n,l > 0$ with $l > i$, the following equation holds in $\widehat{B}_{l+n-1}$:*

$$(e_{i-1} \otimes f_{[1],[n],[0]} \otimes e_{l-i-1}) \cdot f_{[i-1],[n+1],[l-i-1]} = f_{1\cdots i-1,i,i+1\cdots i+n} \cdot f_{[i],[n],[l-i-1]}.$$

Proof. The above equation can be read as

$$(f_{i,i+1\cdots i+n-1,i+n}\cdots f_{i,i+1,i+2})\cdot f_{1\cdots i-1,i\cdots i+n-1,i+n}\cdots f_{1\cdots i-1,i,i+1}$$
$$= f_{1\cdots i-1,i,i+1\cdots i+n}\cdot(f_{1\cdots i,i+1\cdots i+n-1,i+n}\cdots f_{1\cdots i,i+1,i+2}).$$

It can be proved by successive applications of (1.4). □

2. Profinite knots

This section is to present our main results. Our ABC-construction of profinite knots is introduced and the basic properties of profintie knots are shown in §2.1. An absolute Galois action on profinite knots is rigorously established in §2.3, where the notion of pro-l knots introduced in §2.2 serves to show its property.

2.1. **ABC-construction.** Profinite tangles are introduced as consistent finite sequences of symbols of three types A, $\widehat{B}$ and C in Definition 2.2. Profinite links mean profinite tangles without endpoints and profinite knots mean profinite links with a single connected component (Definition 2.3). The notion of isotopy for them are given by a profinite analogue of Turaev moves in Definition 2.7. Two fundamental properties for the set $\widehat{\mathcal{K}}$ of isotopy classes of profinite tangles are presented; Theorem 2.10 explains that there is a natural map from the set $\mathcal{K}$ of isotopy classes of (usual) knots to our set $\widehat{\mathcal{K}}$ and our Theorem 2.14 says that our $\widehat{\mathcal{K}}$ carries a structure of a topological commutative monoid.

The following notion of profinite fundamental tangles plays a role of composite elements of the notion of profinite tangles.

Definition 2.1. The set of *fundamental profinite (oriented)* [2] *tangles* means the disjoint union of the following three sets A, $\widehat{B}$ and C [3] of symbols:

$$A := \left\{a^{\epsilon}_{k,l} \mid k,l=0,1,2,\dots,\ \epsilon=(\epsilon_i)_{i=1}^{k+l+1}\in\{\uparrow,\downarrow\}^k\times\{\curvearrowright,\curvearrowleft\}\times\{\uparrow,\downarrow\}^l\right\},$$
$$\widehat{B} := \left\{b^{\epsilon}_n \mid b^{\epsilon}_n=(b_n,\epsilon=(\epsilon_i)_{i=1}^n)\in\widehat{B}_n\times\{\uparrow,\downarrow\}^n, n=1,2,3,4,\dots\right\},$$
$$C := \left\{c^{\epsilon}_{k,l} \mid k,l=0,1,2,\dots,\ \epsilon=(\epsilon_i)_{i=1}^{k+l+1}\in\{\uparrow,\downarrow\}^k\times\{\circlearrowleft,\circlearrowright\}\times\{\uparrow,\downarrow\}^l\right\}.$$

Here all arrows are merely regarded as symbols.

We occasionally depict these fundamental profinite tangles with ignorance of arrows (which represent orientation of each strings) as the pictures in Figure 2.1, which we call their topological pictures.

For a fundamental profinite tangle γ, its *source* $s(\gamma)$ and its *target* $t(\gamma)$ are sequences of $\uparrow$ and $\downarrow$ defined below:

(1) When $\gamma=a^{\epsilon}_{k,l}$, $s(\gamma)$ is the sequence of $\uparrow$ and $\downarrow$ replacing $\curvearrowright$ (resp. $\curvearrowleft$) by $\uparrow\downarrow$ (resp.$\downarrow\uparrow$) in ϵ and $t(\gamma)$ is the sequence omitting $\curvearrowright$ and $\curvearrowleft$ in ϵ (cf. Figure2.2).
(2) When $\gamma=b^{\epsilon}_n$, $s(\gamma)=\epsilon$ and $t(\gamma)$ is the permutation of ϵ induced by the image of b^{ϵ}_n of the projection $\widehat{B}_n$ to the symmetric group $\mathfrak{S}_n$ (cf. Figure 2.3).

[2] Since during the paper all tangles are assumed to be oriented, we occasionally omit to mention it.

[3] A, B and C stand for Annihilations, Braids and Creations respectively. Our ABC is not related to the ABC-'conjecture'.

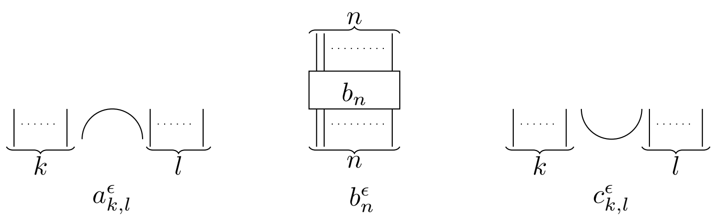


FIGURE 2.1. Topological picture of fundamental profinite tangles

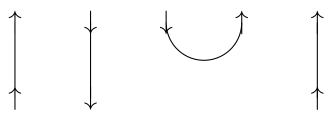


FIGURE 2.2. $a^{\epsilon}_{2,1}$ with $s(a^{\epsilon}_{2,1}) = \uparrow\downarrow\uparrow$ and $t(a^{\epsilon}_{2,1}) = \uparrow\downarrow\downarrow\uparrow\uparrow$

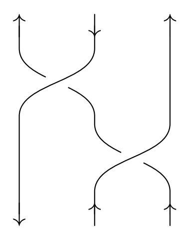


FIGURE 2.3. An example of b^{ϵ}_3 with $s(b^{\epsilon}_3) = \epsilon = \downarrow\uparrow\uparrow$ and $t(b^{\epsilon}_3) = \uparrow\downarrow\uparrow$

(3) When $\gamma = c^{\epsilon}_{k,l}$, $s(\gamma)$ is the set omitting ⤻ and ⤺ in ϵ and $t(\gamma)$ is the set replacing ⤻ (resp. ⤺) by $\downarrow\uparrow$ (resp.$\uparrow\downarrow$) in ϵ.

Definition 2.2. A *profinite (oriented) tangle* means a finite *consistent* [4] sequence $T = \{\gamma_i\}_{i=1}^n$ of fundamental tangles (which we denote by $\gamma_n \cdots \gamma_2 \cdot \gamma_1$). Its source and its target are defined by $s(T) := s(\gamma_1)$ and $t(T) := t(\gamma_n)$. A *profinite (oriented) link* means a profinite tangle T with $s(T) = t(T) = \emptyset$.

For a fundamental profinite tangle γ, its *skeleton* $\mathbb{S}(\gamma)$ is the graph consisting of finite vertices and finite edges connecting them as follows:

(1) When $\gamma = a^{\epsilon}_{k,l}$ or $c^{\epsilon}_{k,l}$, $\mathbb{S}(\gamma)$ is nothing but the graph of the topological picture of γ in Figure 2.1 whose set of vertices is the collection of its endpoints and whose set of edges is given by the arrows connecting them (cf. Figure 2.4).

(2) When $\gamma = b^{\epsilon}_n$, $\mathbb{S}(\gamma)$ is the graph describing the permutation $p(b^{\epsilon}_n)$ (cf. Figure 2.5), where p means the projection $\widehat{B}_n \to \mathfrak{S}_n$. Namely its set of vertices is the collection of its endpoints and its set of edges is the set of diagonal lines combining the corresponding vertices.

[4]Here 'consistent' means that $s(\gamma_{i+1}) = t(\gamma_i)$ holds for all $i = 1, 2, \ldots, n-1$.

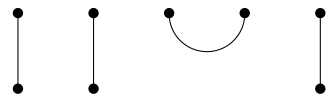


FIGURE 2.4. Skeleton of Figure 2.2

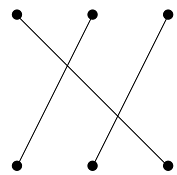


FIGURE 2.5. Skeleton of Figure 2.3

For a profinite tangle $T = \gamma_n \cdots \gamma_2 \cdot \gamma_1$, its skeleton $\mathbb{S}(T) = \mathbb{S}(\gamma_n) \cdots \mathbb{S}(\gamma_2) \cdot \mathbb{S}(\gamma_1)$ means the graph obtained by the composition of $\mathbb{S}(\gamma_i)$ $(1 \leqslant i \leqslant n)$ (cf. Figure 2.6-2.7).

Definition 2.3. A *connected component* of a profinite tangle T means a connected component of the skeleton $\mathbb{S}(T)$ as a graph. A *profinite (oriented) knot* means a profinite (oriented) link with a single connected component.

It is easy to see that the number of connected components of any profinite tangle is always finite. A profinite knot is a profinite version of a (usual) oriented *knot* [5].

Problem 2.4. Is a profinite knot a wild knot[6]?

It might be nice if we could give any topological meaning for all (or a part of) profinite knots.

Example 2.5. The profinite link

$$a_{0,0}^{\frown} \cdot a_{2,0}^{\downarrow\uparrow\frown} \cdot \left(\sigma_2^{\downarrow\uparrow\uparrow\downarrow}\right)^{\lambda} \cdot c_{2,0}^{\downarrow\uparrow\smile} \cdot c_{0,0}^{\smile} \qquad (\lambda \in \widehat{\mathbb{Z}})$$

depicted in Figure 2.6 (here $\sigma_2^{\downarrow\uparrow\uparrow\downarrow}$ is the generator σ_2 of $\widehat{B_4}$) is with 2 connected components if $\lambda \equiv 0 \pmod 2$ and is with a single connected component (hence is a profinite knot) if $\lambda \equiv 1 \pmod 2$. (cf. Figure 2.7).

Notation 2.6. (1) The symbol $e_n^{\epsilon} = (e_n, \epsilon)$ stands for the fundamental profinite tangle in $\widehat{B}$ with $s(e_n^{\epsilon}) = \epsilon$ which corresponds to the trivial braid e_n in $\hat{B}_n$. For a fundamental profinite tangle γ, we mean $e_{n_1}^{\epsilon_1} \otimes \gamma \otimes e_{n_2}^{\epsilon_2}$ by the fundamental profinite tangle obtained by putting $e_{n_1}^{\epsilon_1}$ and $e_{n_2}^{\epsilon_2}$ on the left and on the right of γ respectively. So, $e_{n_1}^{\epsilon_1} \otimes a_{k,l}^{\epsilon} \otimes e_{n_2}^{\epsilon_2} = a_{n_1+k,l+n_2}^{\epsilon_1,\epsilon,\epsilon_2}$, for instance. For a profinite tangle $T = \{\gamma_i\}_{i=1}^n$ (γ_i: a fundamental profinite tangle), $e_{n_1}^{\epsilon_1} \otimes T \otimes e_{n_2}^{\epsilon_2}$ means the profinite tangle $\{e_{n_1}^{\epsilon_1} \otimes \gamma_i \otimes e_{n_2}^{\epsilon_2}\}_{i=1}^n$.

(2) Let $\epsilon_0 \in \{\uparrow, \downarrow\}^n$ for some n. For a fundamental profinite tangle $b_l^{\epsilon} \in \widehat{B}$ and k with $1 \leqslant k \leqslant l$, the symbol $ev_{k,\epsilon_0}(b_l^{\epsilon})$ (resp. $ev^{k,\epsilon_0}(b_l^{\epsilon})$) means the

[5] A *knot* means a smooth embedding of the oriented circle into the three dimensional sphere S^3 (or $\mathbb{R}^3$).

[6] A *wild knot* means a *topological* embedding of the oriented circle into S^3 (or $\mathbb{R}^3$).

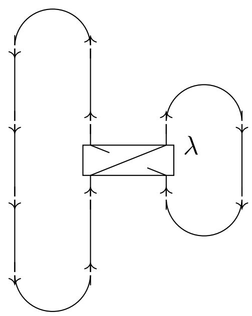
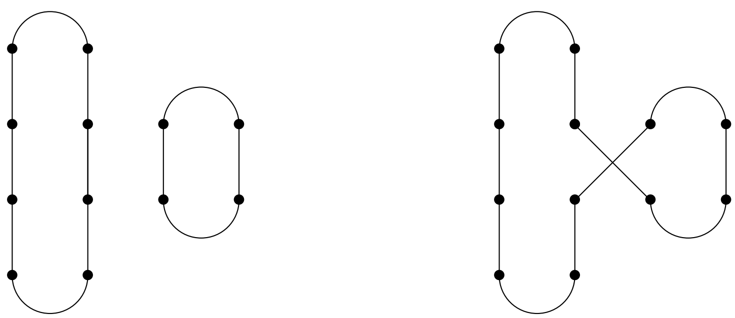


FIGURE 2.6. Is this a profinite knot?



$\lambda \equiv 0 \pmod 2$ $\lambda \equiv 1 \pmod 2$

FIGURE 2.7. Skeleton of Figure 2.6

element in $\widehat{B}$ which represents the profinite braids replacing the k-th string (from its bottom (resp. above) left) of b_l^ϵ by the trivial braid $e_n^{\epsilon_0}$ with n strings whose source is ϵ_0 (cf. Notation 1.17). For instance, the profinite tangle in Figure 2.3 is described as $ev_{1,\downarrow\uparrow}\big(\ \big)$ or $ev^{2,\downarrow\uparrow}\big(\ \big)$.

Definition 2.7. For profinite tangles, the moves (T1)-(T6) are defined as follow.

(T1) *Trivial braids invariance*: for a profinite tangle T with $|s(T)| = m$ (resp. $|t(T)| = n$), [7]

$$e_n^{t(T)} \cdot T = T = T \cdot e_m^{s(T)}.$$

For e_n, see Notation 2.6. Figure 2.8 depicts the move.

(T2) *Braids composition*: for $b_n^{\epsilon_1}, b_n^{\epsilon_2} \in \widehat{B}$ with $t(b_n^{\epsilon_1}) = s(b_n^{\epsilon_2})$,

$$b_n^{\epsilon_2} \cdot b_n^{\epsilon_1} = b_n^{\epsilon_3}.$$

Here $b_n^{\epsilon_3}$ means the element in $\widehat{B}$ with $s(b_n^{\epsilon_3}) = s(b_n^{\epsilon_1})$ and $t(b_n^{\epsilon_3}) = t(b_n^{\epsilon_2})$ which represents the product $b_2 \cdot b_1$ of two braids in $\widehat{B}_n$. Figure 2.9 depicts the move.

(T3) *Independent tangles relation*: for profinite tangles T_1 and T_2 with $|s(T_1)| = m_1$, $|t(T_1)| = n_1$, $|s(T_2)| = m_2$ and $|t(T_2)| = n_2$,

$$(e_{n_1}^{t(T_1)} \otimes T_2) \cdot (T_1 \otimes e_{m_2}^{s(T_2)}) = (T_1 \otimes e_{n_2}^{t(T_2)}) \cdot (e_{m_1}^{s(T_1)} \otimes T_2).$$

[7] For a set S, $|S|$ stands for its cardinality.

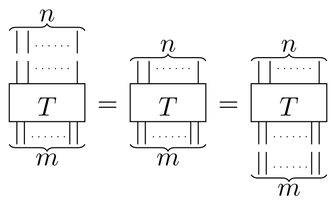


FIGURE 2.8. (T1): Trivial braids invariance

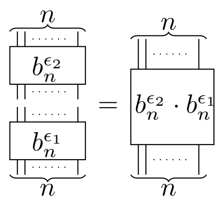


FIGURE 2.9. (T2): braids composition

We occasionally denote both hands side by $T_1 \otimes T_2$. For the symbol $\otimes$, see Notation 2.6. Figure 2.10 depicts the move.

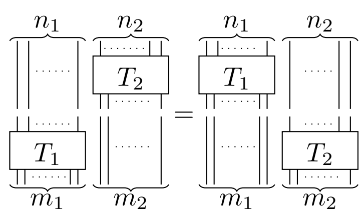


FIGURE 2.10. (T3): independent tangles relation

(T4) *Braid-tangle relations*: for $b_l^\epsilon \in \widehat{B}$, k with $1 \leqslant k \leqslant l$ and a profinite tangle T with $|s(T)| = m$ and $|t(T)| = n$,

$$ev_{k,t(T)}(b_l^\epsilon) \cdot (e_{k-1}^{s_1} \otimes T \otimes e_{l-k}^{s_2}) = (e_{k'-1}^{t_1} \otimes T \otimes e_{l-k'}^{t_2}) \cdot ev^{k',s(T)}(b_l^\epsilon).$$

For ev, see Notation 2.6. For $s(b_l^\epsilon) = \epsilon = (\epsilon_i)_{i=1}^l$ we put $s_1 := (\epsilon_i)_{i=1}^{k-1}$ and $s_2 := (\epsilon_i)_{i=k+1}^l$. Put $k' = b_l^\epsilon(k)$. Here $b_l^\epsilon(k)$ stands for the image of k by the permutation which corresponds to b_l^ϵ by the projection $B_l \to \mathfrak{S}_l$. For $t(b_l^\epsilon) = (\epsilon'_i)_{i=1}^l$ we put $t_1 := (\epsilon'_i)_{i=1}^{k'-1}$ and $t_2 := (\epsilon'_i)_{i=k'+1}^l$. Figure 2.11 depicts the move.

(T5) *Creation-annihilation relation*: for $c_{k,l}^\epsilon \in C$ and $a_{k+1,l-1}^{\epsilon'} \in A$ with $t(c_{k,l}^\epsilon) = s(a_{k+1,l-1}^{\epsilon'})$

$$a_{k+1,l-1}^{\epsilon'} \cdot c_{k,l}^\epsilon = e_{k+l}^{s(c_{k,l}^\epsilon)}.$$

And for $c_{k,l}^\epsilon \in C$ and $a_{k-1,l+1}^{\epsilon'} \in A$ with $t(c_{k,l}^\epsilon) = s(a_{k-1,l+1}^{\epsilon'})$

$$a_{k-1,l+1}^{\epsilon'} \cdot c_{k,l}^\epsilon = e_{k+l}^{s(c_{k,l}^\epsilon)}.$$

Figure 2.12 depicts the move.

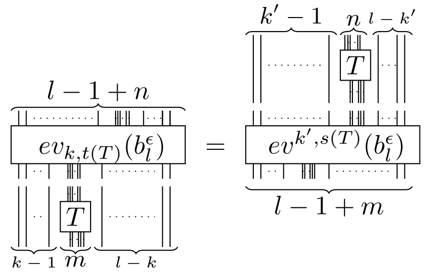


FIGURE 2.11. (T4): braid-tangle relation

FIGURE 2.12. (T5): creation-annihilation relations

(T6) *Creation-commutativity and commutativity-annihilation relations*: for $c \in \widehat{\mathbb{Z}}$, $c^\epsilon_{k,l} \in C$ and $\sigma^{\epsilon'}_{k+1} \in \widehat{B}$ which represents $\sigma_{k+1} \in \widehat{B}_{k+l+2}$ and $t(c^\epsilon_{k,l}) = s(\sigma^{\epsilon'}_{k+1})$

$$(\sigma^{\epsilon'}_{k+1})^{2c+1} \cdot c^\epsilon_{k,l} = c^{\bar\epsilon}_{k,l}$$

where $\bar\epsilon$ indicates the switch of ↶ and ↷ in ϵ.

And for $c \in \widehat{\mathbb{Z}}$ [8], $a^\epsilon_{k,l} \in A$ and $\sigma^{\epsilon'}_{k+1} \in \widehat{B}$ which represents $\sigma_{k+1} \in \widehat{B}_{k+l+2}$ and $s(a^\epsilon_{k,l}) = t(\sigma^{\epsilon'}_{k+1})$

$$a^\epsilon_{k,l} \cdot (\sigma^{\epsilon'}_{k+1})^{2c+1} = a^{\bar\epsilon}_{k,l}.$$

where $\bar\epsilon$ again indicates the switch of ↷ and ↶ in ϵ. Figure 2.13 depicts the move.

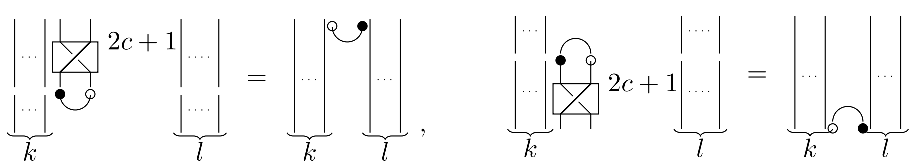


FIGURE 2.13. (T6): creation-commutativity/commutativity-annihilation relation

These moves (T1)-(T6) are profinite analogues of the so-called *Turaev moves* [Tu] for oriented tangles (consult also [CDM, K, O1]). Our above formulation is stimulated by the moves presented in [Ba] (R1)-(R11).

Definition 2.8. Two profinite (oriented) tangles T_1 and T_2 are *isotopic*, denoted $T_1 \sim T_2$, if they are related by a *finite* number of the moves (T1)-(T6). *The set of isotopy classes of oriented profinite tangles* is denoted by $\widehat{\mathcal{T}}$. *The set of isotopy class* $\widehat{\mathcal{K}}$ *of profinite knots* is the subset of $\widehat{\mathcal{T}}$ which consists of isotopy classes of profinite knots.

[8] It should be worthy to emphasize that c is assumed to be not in $\mathbb{Z}$ but in $\widehat{\mathbb{Z}}$.

Note 2.9. Profinite topology on $\widehat{B}_n$ $(n = 1, 2, \dots)$ and the discrete topology on A and on C yield a topology on the space of profinite tangles. Hence $\widehat{\mathcal{T}}$ carries a structure of topological space.

Theorem 2.10. *(1). Let $\mathcal{T}$ be the set of isotopy classes of (usual) oriented tangles.* [9] *There is a natural map*

$$h : \mathcal{T} \to \widehat{\mathcal{T}},$$

which we call the profinite realization map.

(2). The above profinite realization map induces the map

$$h : \mathcal{K} \to \widehat{\mathcal{K}}.$$

Here $\mathcal{K}$ stands for the set of isotopy classes of usual oriented knots.

Proof. (1). The result in [Ba] indicates that the set $\mathcal{T}$ is described by the set of consistent finite sequences of fundamental tangles, elements of A,

$$B := \left\{ b_n^\epsilon \;\middle|\; b_n^\epsilon = (b_n, \epsilon = (\epsilon_i)_{i=1}^n) \in B_n \times \{\uparrow, \downarrow\}^n, n = 1, 2, 3, 4, \dots \right\}$$

and C, modulo the (usual) Turaev moves. Since the *(usual) Turaev moves* in this case mean the moves replacing profinite tangles and braids by (discrete) tangles and braids in (T1)-(T6) and $c \in \widehat{\mathbb{Z}}$ by $c \in \mathbb{Z}$ in (T6). Because we have a natural map $B_n \to \widehat{B}_n$ and the Turaev moves are special case of our 6 moves, we have a natural map $h : \mathcal{T} \to \widehat{\mathcal{T}}$.

(2). It is easy because the set of profinite tangles isotopic to a given profinite knot consists of only profinite knots. □

We notice that the number of connected components is an isotopic invariant of profinite tangles. As a knot analogue of residually-finiteness (1.1) of the braid group B_n, we raise the conjecture below.

Conjecture 2.11. The map $h : \mathcal{K} \to \widehat{\mathcal{K}}$ is injective.

Remark 2.12. If the injectivity of h failed, then the Kontsevich invariant [Ko] would fail to be a perfect invariant by the arguments given in Remark 2.29 below. We remind that the Kontsevich invariant is an invariant of oriented knots which is conjectured to be a perfect invariant, i.e. an invariant detecting all oriented knots.

Extending various known knot invariants into profinite knots is one of our fundamental problems. Particularly unknotting number (cf. say, [CDM]) looks mysterious. A profinite analogue of the Alexander polynomial, that of the Jones polynomial, etc, will be introduced and discussed in our subsequent paper [F2].

Below we remind one of the most elementary results for oriented knots.

Proposition 2.13. *The space $\mathcal{K}$ of usual oriented knots carries a structure of a commutative associative monoid by the connected sum (knot sum).*

Here the connected sum (knot sum) is a natural way to fuse two oriented knots, with an appropriate position of orientation, into one (an example is illustrated in Figure 2.14).

---

[9] A *tangle* means a finite disjoint union of embedded one-dimensional intervals and circles in $\mathbb{R}^2 \times [0, 1]$. It is like 'a braid' whose each connected component is allowed to be a circle and have endpoints on the same plane. For precise, consult standard textbooks such as [CDM, O1]

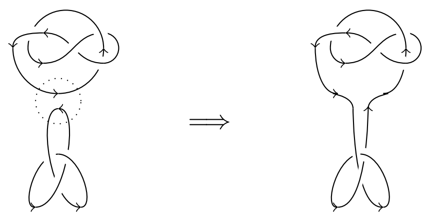


FIGURE 2.14. Connected sum (knot sum)

It can be done any points. It is actually well-defined. It yields a commutative monoid structure on $\mathcal{K}$. For more, consult the standard text book of knot theory.

The notion of connected sum can be extended into profinite knots.

Theorem 2.14. *For any two profinite knots $K_1 = \alpha_m \cdots \alpha_1$ and $K_2 = \beta_n \cdots \beta_1$ with $(\alpha_m, \alpha_1) = (\curvearrowleft, \smile)$ and $(\beta_n, \beta_1) = (\curvearrowleft, \smile)$, their* connected sum *means the profinite tangle defined by*

$$K_1 \sharp K_2 := \alpha_m \cdots \alpha_2 \cdot \beta_{n-1} \cdots \beta_1. \tag{2.1}$$

Then

(1). the above connected sum induces a well-defined product

$$\sharp : \widehat{\mathcal{K}} \times \widehat{\mathcal{K}} \to \widehat{\mathcal{K}}.$$

(2). By the product $\sharp$, the set $\widehat{\mathcal{K}}$ forms a topological (that is, the map $\sharp$ is continuous with respect to the topology given above) commutative associative monoid, whose unit is given by the oriented circle $\circlearrowleft := \curvearrowleft \cdot \smile$.

(3). The profinite realization map $h : \mathcal{K} \to \widehat{\mathcal{K}}$ forms a monoid homomorphism whose image is dense in $\widehat{\mathcal{K}}$.

Proof. (1). Since each isotopy class of profinite knot contains by (T6) a profinite knot K of the above type; a profinite knot starting with $\smile$ and ending at $\curvearrowleft$, we can show that the connected sum extends to $\widehat{\mathcal{K}}$ once we have the well-definedness.

Firstly we prove that $K_1 \sharp K_2$ is isotopic to $K_1' \sharp K_2$ if K_1' is isotopic to K_1, i.e., K_1' is obtained by a finite number of our moves (T1)-(T6) from K_1. We may assume that K_1' is obtained from K_1 by a single operation of one of the 6 moves. In the case when this move effects only on α_i's for $i > 1$, it is easy to see our claim. If the moves effects on α_1, it must be (T3) or (T6). Consider the latter case (T6). It suffices to show that $K_1 \sharp K_2$ is isotopic to $K_3 = \alpha_m \cdots \alpha_2 \cdot \sigma^{2c} \cdot \beta_{n-1} \cdots \beta_1$ for $c \in \widehat{\mathbb{Z}}$. The proof is depicted in Figure 2.15. Here $S_1 = \alpha_{m-1} \cdots \alpha_2$ and $S_2 = \beta_n \cdots \beta_2$. We note that the first and the fourth equalities follow from (T3) and (T5). We use (T4) in the second equality for σ^{2c} and the dashed box. We derived the third equality from (T6).

Next consider the former case (T3). Since K_1 is a profinite knot, m_1 and m_2 are both 0. By the above argument in case (T6), we may assume that both T_1 and T_2 in Figure 2.10 should start from $\smile$ (i.e. $\alpha_1 = \beta_1 = \smile$). Define T as in Figure 2.16. A successive application of commutativity of profinite braids with T shown in (T4) and that of creations and annihilations with T shown in Lemma 2.16 lead the isotopy equivalence shown in Figure 2.17. Here T_1' and T_2' stand for emissions of the lowest $\smile$ from T_1 and T_2. We note that emission of T in the figure represents K_1 and K_1'.

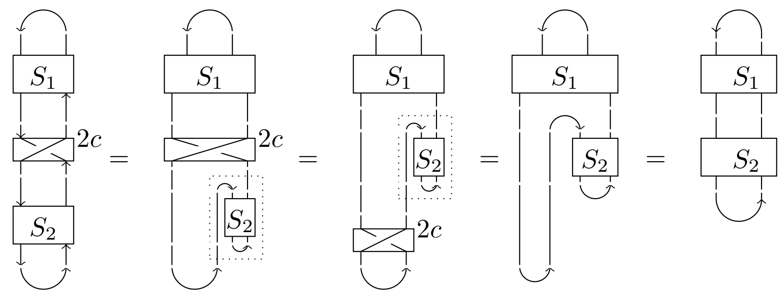


FIGURE 2.15. $K_3 = K_1 \sharp K_2$

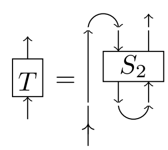


FIGURE 2.16. definition of T

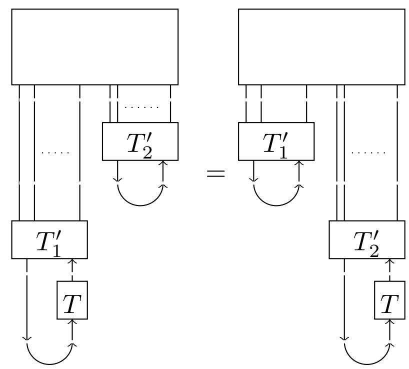


FIGURE 2.17. $K_1 \sharp K_2 = K_1' \sharp K_2$

Secondly we must prove that $K_1 \sharp K_2$ is isotopic to $K_1 \sharp K_2'$ if K_2' is isotopic to K_2. It can be proved in a completely same way to the above arguments. Thus our proof is finally completed.

(2). Associativity, i.e. $(K_1 \sharp K_2) \sharp K_3 = K_1 \sharp (K_2 \sharp K_3)$, is easy to see. A proof of commutativity is illustrated in Figure 2.18. We note that we use (T3) and (T5) in the first, the third and the fifth equalities and we apply (T4) and Lemma 2.16 for the dashed boxes in the second and the fourth equalities.

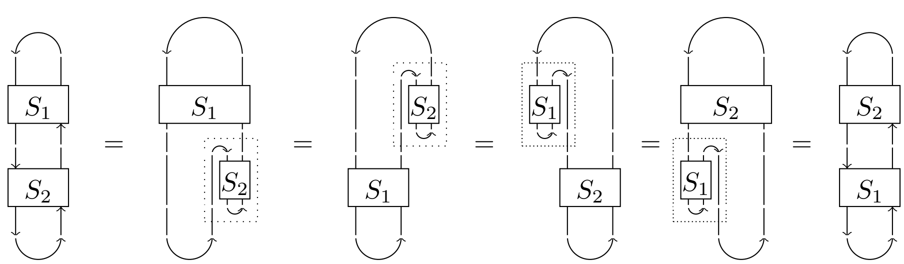


FIGURE 2.18. $K_1 \sharp K_2 = K_2 \sharp K_1$

To show that $\sharp$ is continuous, we define by $\widehat{\mathcal{T}}^{\mathrm{seq}}$ the set of finite consistent sequences of profinite fundamental tangles and by $\widehat{\mathcal{K}'}^{\mathrm{seq}}$ the set of finite consistent

sequences of profinite fundamental tangles $\gamma_n \cdots \gamma_2 \cdot \gamma_1$ with a single connected component and with $(\gamma_n, \gamma_1) = (\curvearrowleft, \curvearrowright)$. We note that the quotient set of $\widehat{\mathcal{T}}^{\mathrm{seq}}$ by the equivalence of finite sequences of the moves (T1)-(T6) is equal to the set $\widehat{\mathcal{T}}$ of profinite tangles. We also note that $\widehat{\mathcal{K}'}^{\mathrm{seq}}$ is projected onto its subset $\widehat{\mathcal{K}}$ of profinite knots. The set $\widehat{\mathcal{T}}^{\mathrm{seq}}$ carries a structure of a topological space by the profinite topologies on $\widehat{B}_n$ (Note 2.9). It induces a subspace topology on $\widehat{\mathcal{K}'}^{\mathrm{seq}}$. The map $\widehat{\mathcal{T}}^{\mathrm{seq}} \twoheadrightarrow \widehat{\mathcal{T}}$ is continuous, hence so the projection $\widehat{\mathcal{K}'}^{\mathrm{seq}} \twoheadrightarrow \widehat{\mathcal{K}}$ is. By the topology it is easy to see that the map

$$\sharp : \widehat{\mathcal{K}'}^{\mathrm{seq}} \times \widehat{\mathcal{K}'}^{\mathrm{seq}} \to \widehat{\mathcal{K}'}^{\mathrm{seq}}$$

caused by (2.1) is continuous. Because the following diagram is commutative

$$\begin{tikzcd}
\widehat{\mathcal{K}'}^{\mathrm{seq}} \times \widehat{\mathcal{K}'}^{\mathrm{seq}} \arrow[r, "\sharp"] \arrow[d] & \widehat{\mathcal{K}'}^{\mathrm{seq}} \arrow[d] \\
\widehat{\mathcal{K}} \times \widehat{\mathcal{K}} \arrow[r, "\sharp"] & \widehat{\mathcal{K}}
\end{tikzcd}$$

and the projection $\widehat{\mathcal{K}'}^{\mathrm{seq}} \twoheadrightarrow \widehat{\mathcal{K}}$ is continuous, the lower map is also continuous.

(3). The first statement is obvious. Let $\mathcal{K}'^{\mathrm{seq}}$ be the subset of $\widehat{\mathcal{K}'}^{\mathrm{seq}}$ which consists of consistent finite sequences of 'usual fundamental tangles', that is, sequences of elements in A, C and B with $b_n \in B_n \subset \widehat{B}_n$. There are a natural inclusion $\mathcal{K}'^{\mathrm{seq}} \to \widehat{\mathcal{K}'}^{\mathrm{seq}}$ and a surjection $\mathcal{K}'^{\mathrm{seq}} \to \mathcal{K}$. Since the map $B_n \to \widehat{B}_n$ is with dense image, so the inclusion $\mathcal{K}'^{\mathrm{seq}} \to \widehat{\mathcal{K}'}^{\mathrm{seq}}$ is. Therefore the map $\mathcal{K} \to \widehat{\mathcal{K}}$ is also with dense image. It is because the following diagram is commutative

$$\begin{tikzcd}
\mathcal{K}'^{\mathrm{seq}} \arrow[r] \arrow[d] & \widehat{\mathcal{K}'}^{\mathrm{seq}} \arrow[d] \\
\mathcal{K} \arrow[r] & \widehat{\mathcal{K}}
\end{tikzcd}$$

and the projection $\widehat{\mathcal{K}'}^{\mathrm{seq}} \twoheadrightarrow \widehat{\mathcal{K}}$ is continuous. $\square$

The following two lemmas are required to prove Theorem 2.14 (1).

Lemma 2.15. *For a profinite tangle T with $s(T) = t(T) = \uparrow$, define its transpose $\rotatebox{180}{$T$}$ (occasionally also denoted by ${}^tT$) by*

$$\rotatebox{180}{T} := a_{0,1}^{\frown\downarrow} \cdot (e_1^{\downarrow} \otimes T \otimes e_1^{\downarrow}) \cdot c_{1,0}^{\downarrow\smile}.$$

Then we have

$$\rotatebox{180}{T} = a_{1,0}^{\downarrow\frown} \cdot (e_1^{\downarrow} \otimes T \otimes e_1^{\downarrow}) \cdot c_{0,1}^{\smile\downarrow}.$$

Similar claim holds for a profinite tangle T with $s(T) = t(T) = \downarrow$ by reversing all arrows.

Figure 2.19 describes our lemma.

Proof. A proof is depicted in Figure 2.20. We note that we apply (T1) and (T4) for the dashed boxes in the first and the fourth equalities and we use (T2), (T3) and (T6) in the second and the the third equalities. $\square$

By (T5) and the above lemma, we see that ${}^{tt}T$ is isotopic to T.

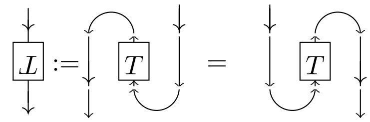


FIGURE 2.19. transpose of T

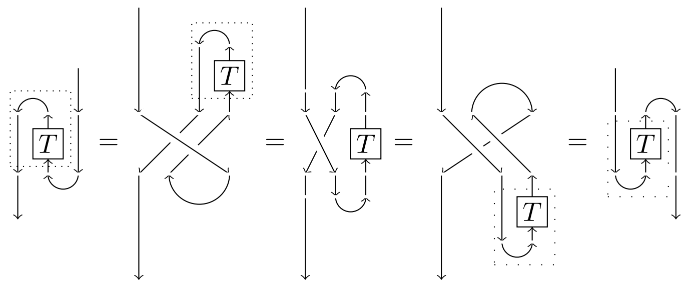


FIGURE 2.20. proof of Lemma 2.15

Lemma 2.16. *For a profinite tangle T with $s(T) = t(T) =\uparrow$, the equalities in Figure 2.21 hold. The same claim also holds for a profinite tangle T with $s(T) = t(T) =\downarrow$ by reversing all arrows.*

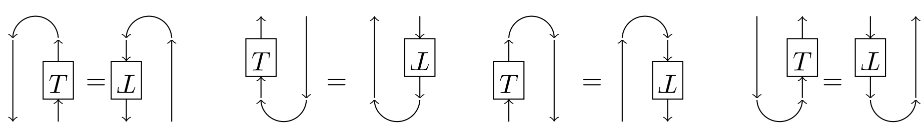


FIGURE 2.21. Creation and annihilation commute with T.

Proof. It can be checked by direct computation using Lemma 2.15. □

Remark 2.17. The so-called *Alexander-Markov's theorem* claims a one-to-one correspondence

$$\mathcal{L} \longleftrightarrow \sqcup_n B_n / \sim_{\mathrm{M}}$$

between the set $\mathcal{L}$ of isotopy classes of oriented links and the (disjoint) union $\sqcup_n B_n$ of braids groups modulo the equivalence $\sim_{\mathrm{M}}$ given by the Markov moves

(M1). $b_1{\cdot}b_2 \sim_{\mathrm{M}} b_2{\cdot}b_1 \quad (b_1, b_2 \in B_n)$, (M2). $b \in B_n \sim_{\mathrm{M}} b\sigma_n^{\pm 1} \in B_{n+1} \quad (b \in B_n)$

(consult [CDM, O1] for example).

The question below is to ask a validity of profinite analogue of Alexander-Markov's theorem.

Problem 2.18. Is there a 'profinite analogue' of the Alexander-Markov's theorem which holds for the set $\widehat{\mathcal{L}}$ of isotopy classes of profinite links ?

There are several proofs of Alexander-Markov's theorem for usual links ([Bi, Tr, V, Y] etc). But they look heavily based on a certain finiteness property, which we (at least the author) may not expect the validity for profinite links.

2.2. **Pro-l knots.** Pro-l tangles, pro-l knots and isotopy among them which are pro-l analogues of our corresponding notions given in the previous subsection, are introduced in Definition 2.23. A natural map from profinite tangles (resp. profinite knots) to pro-l tangles (resp. pro-l knots) is constructed in Proposition 2.24. Proalgebraic tangles and proalgebraic knots are given in Definition 2.27. There is a natural map from pro-l tangles (resp. pro-l knots) to proalgebraic tangles (resp. proalgebraic knots) in Proposition 2.28. It is explained that the Kontsevich invariant factors through these natural maps in Remark 2.29. Our consideration in this subsection will serve for a proof of non-triviality of the Galois action constructed in §2.3.

Let l be a prime. We may include $l=2$.

Notation 2.19. A topological group G is called a *pro-l group* if it is a projective limit $\varprojlim G_i$ of a projective system of finite l-groups $\{G_i\}_{i\in I}$. For a discrete group Γ, its *pro-l completion* $\widehat{\Gamma}^l$ is the pro-l group defined by the projective limit

$$\widehat{\Gamma}^l=\varprojlim \Gamma/N$$

where N runs over all normal subgroups of Γ with finite indices of power of l.

For more on pro-l groups, consult [RZ] for example. We note there is a natural homomorphism $\Gamma\to\widehat{\Gamma}^l$.

Notation 2.20. Let $n\geqslant 2$. Let $\widehat{P}_n\rtimes B_n$ be the semi-direct product of $\widehat{P}_n$ and B_n with respect to the B_n-action on $\widehat{P}_n$ given by $p\mapsto bpb^{-1}$ ($p\in\widehat{P}_n$ and $b\in B_n$). consider the inclusion $P_n\hookrightarrow\widehat{P}_n\rtimes B_n$ sending $p\mapsto(p,p^{-1})$. Then it is easy to see the homomorphism sending $(p,b)\mapsto pb$ yields an isomorphism:

$$(\widehat{P}_n\rtimes B_n)/P_n\simeq\widehat{B}_n.$$

Definition 2.21. (1). The *pro-l pure braid group* $\widehat{P}^l_n$ is the pro-l completion of P_n.

(2). The *pro-(l) braid group* $\widehat{B}^{(l)}_n$ is defined to be the induced quotient

$$\widehat{B}^{(l)}_n:=(\widehat{P}^l_n\rtimes B_n)/P_n.$$

We encode a topological group structure on $\widehat{B}^{(l)}_n$ by the pro-l topology on $\widehat{P}^l_n$ and the discrete topology on B_n. We note that this $\widehat{B}^{(l)}_n$ appears also in [LS].

Remark 2.22. (1). Our $\widehat{B}^{(l)}_n$ is different from the pro-l completion $\widehat{B}^l_n$ of B_n.

(2). There is an exact sequence:

$$1\to\widehat{P}^l_n\to\widehat{B}^{(l)}_n\to\mathfrak{S}_n\to 1.$$

(3). There are natural group homomorphisms:

$$B_n\to\widehat{B}_n\to\widehat{B}^{(l)}_n. \tag{2.2}$$

The map (2.2) is induced from $P_n\to\widehat{P}_n\to\widehat{P}^l_n$.

Definition 2.23. (1). A *pro-l tangle* means a consistent finite sequence of *fundamental pro-l tangles*, which are elements in A, C (in Definition 2.1) or

$$\widehat{B}^l:=\left\{b^\epsilon_n\ \middle|\ b^\epsilon_n=(b_n,\epsilon=(\epsilon_i)^n_{i=1})\in\widehat{B}^{(l)}_n\times\{\uparrow,\downarrow\}^n, n=1,2,3,4,\dots\right\}.$$

A *pro-l knot* means a pro-l tangle without endpoints (their sources and targets are both empty) and with a single connected component.

(2). Two pro-l tangles T_1 and T_2 are called *isotopic* if they are related by a finite number of the moves replacing profinite tangles and profinite braids by pro-l tangles and elements in $\widehat{B}^l$ in (T1)-(T6) and $c \in \widehat{\mathbb{Z}}$ by $c \in \mathbb{Z}_l$ in (T6). [10]

We denote the set of isotopy classes of pro-l tangles by $\widehat{\mathcal{T}}^l$ and the set of isotopy classes of pro-l knots by $\widehat{\mathcal{K}}^l$. Both spaces carry a structure of topological space by the method in Note 2.9.

Proposition 2.24. *(1). The set $\widehat{\mathcal{K}}^l$ forms a topological monoid with respect to the connected sum.*

(2). There are continuous maps:

$$\widehat{\mathcal{T}} \to \widehat{\mathcal{T}}^l, \tag{2.3}$$

$$\widehat{\mathcal{K}} \to \widehat{\mathcal{K}}^l. \tag{2.4}$$

(3). The map (2.4) *is monoid homomorphisms. The image of its composition with h in Theorem 2.10.(2)*

$$h_l : \mathcal{K} \xrightarrow{h} \widehat{\mathcal{K}} \to \widehat{\mathcal{K}}^l. \tag{2.5}$$

is with dense image in $\widehat{\mathcal{K}}^l$.

Proof. (1). It is obtained by the same arguments to the proof of Theorem 2.14.

(2). The map (2.3) is induced from the second map in (2.2), whose continuity implies ours. It preserves each connected component, which yields the map (2.4).

(3). To see that they form homomorphisms are immediate. The density can be proved by the same arguments to the proof of Theorem 2.14. □

It is direct to see that the maps $\mathcal{T} \xrightarrow{h} \widehat{\mathcal{T}}$ and $\mathcal{K} \xrightarrow{h} \widehat{\mathcal{K}}$ above are both equal to the maps in Theorem 2.10 denoted by the same symbol h.

Definition 2.25. Let R be a commutative ring.

(1). Let I be the two-sided ideal of the group algebra $R[B_n]$ of B_n generated by $\sigma_i - \sigma_i^{-1}$ $(1 \leqslant i \leqslant n-1)$. The topological R-algebra $\widehat{R[B_n]}$ of *proalgebraic braids* means its completion with respect to the I-adic filtration, i.e.

$$\widehat{R[B_n]} := \varprojlim_N R[B_n]/I^N.$$

(2). Put $I_0 := I \cap R[P_n]$. Then I_0 is the augmentation ideal of $R[P_n]$ (cf. [KT]). The topological R-algebra $\widehat{R[P_n]}$ of *proalgebraic pure braids* means its completion

$$\widehat{R[P_n]} := \varprojlim_N R[P_n]/I_0^N.$$

It is a subalgebra of $\widehat{R[B_n]}$.

Actually both algebras equip a structure of co-commutative Hopf algebra.

Since $\widehat{P}_n^l$ is a pro-l group and I_0 is the augmentation ideal, we have a natural continuous homomorphism $\widehat{P}_n^l \to \widehat{\mathbb{Q}_l[P_n]}$ (cf. [CDM] for example).

[10] We note that, for $\sigma_i \in \widehat{B}_N^{(l)}$ and $c \in \mathbb{Z}_l$, the power σ_i^{2c+1} makes sense in $\widehat{B}_N^{(l)}$ because, by $\sigma_i^2 \in \widehat{P}_N^l$, we have $\sigma_i^{2c} \in \widehat{P}_N^l$.

Proposition 2.26. *There is a natural continuous group homomorphism*

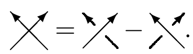
$$\widehat{B}_n^{(l)} \to \widehat{\mathbb{Q}_l[B_n]}. \tag{2.6}$$

Proof. It can be directly checked that the map induced from the above $\widehat{P}_n^l \to \widehat{\mathbb{Q}_l[P_n]}$ ($\subset \widehat{\mathbb{Q}_l[B_n]}$) and the natural map $B_n \hookrightarrow \mathbb{Q}_l[B_n] \to \widehat{\mathbb{Q}_l[B_n]}$ holds the property. $\square$

Next we discuss the corresponding notions in tangles and knots settings. The following notions go back to the idea of Vassiliev.

Definition 2.27 ([KT])**.** Let R be a commutative ring.

(1). Let $R[\mathcal{T}]$ be the free R-module of finite formal sums of elements of $\mathcal{T}$. A singular oriented tangle [11] determines an element of $R[\mathcal{T}]$ by the desingularization of each double point by the following relation



Let $\mathcal{T}_n$ ($n \geqslant 0$) be the R-submodule of $R[\mathcal{T}]$ generated by all singular oriented tangles with n double points. The descending filtration $\{\mathcal{T}_n\}_{n\geqslant 0}$ is called the *singular filtration*. The topological R-module $\widehat{R[\mathcal{T}]}$ of *proalgebraic tangles* means its completion with respect to the singular filtration:

$$\widehat{R[\mathcal{T}]} := \varprojlim_N R[\mathcal{T}]/\mathcal{T}_N.$$

(2). Let $R[\mathcal{K}]$ be the R-submodule of $R[\mathcal{T}]$ generated by elements of $\mathcal{K}$. By Proposition 2.13, it forms a commutative R-algebra. Put $\mathcal{K}_n := \mathcal{T}_n \cap R[\mathcal{K}]$ ($n \geqslant 0$). Then $\mathcal{K}_n$ forms an ideal of $R[\mathcal{K}]$ and the descending filtration $\{\mathcal{K}_n\}_{n\geqslant 0}$ is called the *singular knot filtration* (cf. loc.cit.). The topological commutative R-algebra $\widehat{R[\mathcal{K}]}$ of *proalgebraic knots* means its completion with respect to the singular knot filtration:

$$\widehat{R[\mathcal{K}]} := \varprojlim_N R[\mathcal{K}]/\mathcal{K}_N.$$

Actually it equips a structure of co-commutative and commutative Hopf algebra.

The maps below are tangle and knot analogues of the map (2.6).

Proposition 2.28. *(1). There are continuous maps:*

$$\widehat{\mathcal{T}}^l \to \widehat{\mathbb{Q}_l[\mathcal{T}]}, \tag{2.7}$$

$$\widehat{\mathcal{K}}^l \to \widehat{\mathbb{Q}_l[\mathcal{K}]}. \tag{2.8}$$

(2). The map (2.8) *is a continuous monoid homomorphism and its image lies on the set* $\widehat{\mathbb{Q}_l[\mathcal{K}]}^{\times}$ *of invertible elements.*

Proof. (1). Since an element $b_n^\epsilon \in B_n \times \{\uparrow, \downarrow\}^n$ ($n \geqslant 1$), a braid $b_n \in B_n$ with an orientation $\epsilon \in \{\uparrow, \downarrow\}^n$ (namely its source), is naturally regarded as a special type of an oriented tangle, each orientation ϵ yields a natural inclusion

$$\mathbb{Q}_l[B_n] \hookrightarrow \mathbb{Q}_l[\mathcal{T}].$$

[11] It is an 'oriented tangle' which is allowed to have a finite number of transversal double points (see [KT] for precise).

On the embedding, we have $\mathcal{T}_m \cap \mathbb{Q}_l[B_n] = I^m$ for $m \geqslant 0$. Therefore the above map and the map (2.6) induce

$$\widehat{B}_n^{(l)} \to \mathbb{Q}_l[\mathcal{T}]/\mathcal{T}_m.$$

Hence it determines the map of sets

$$\widehat{B}^l \to \mathbb{Q}_l[\mathcal{T}]/\mathcal{T}_m. \tag{2.9}$$

We also have the natural maps of sets

$$A \to \mathbb{Q}_l[\mathcal{T}]/\mathcal{T}_m \text{ and } C \to \mathbb{Q}_l[\mathcal{T}]/\mathcal{T}_m. \tag{2.10}$$

As is described in the proof of Theorem 2.10, the set $\mathcal{T}$ of (usual) oriented tangles is described by the set of consistent finite sequences of elements of A, B and C modulo the (usual) Turaev moves. By $\mathbb{Q}_l$-linearly extending the description, we obtain the same description of $\mathbb{Q}_l[\mathcal{T}]$. Since our three maps (2.9) and (2.10) are consistent with the moves, we obtain

$$\widehat{\mathcal{T}}^l \to \mathbb{Q}_l[\mathcal{T}]/\mathcal{T}_m.$$

(Again we note that, for $\sigma_i \in B_N$ and $c \in \mathbb{Z}_l$, the power σ_i^{2c+1} makes sense in $\mathbb{Q}_l[B_N]/I^m$ by the formula

$$\sigma_i^{2c+1} := \sigma_i \cdot \exp\{c \log \sigma_i^2\}.$$

Here exp and log are defined by the usual Taylor expansions. The RHS is well-defined by $\sigma_i^2 - 1 \in I$.)

It yields the map (2.7) which is continuous. Since this map preserves each connected component, the map (2.8) is also obtained.

(2). It is immediate to see that it forms a continuous homomorphism.

Each oriented knot, an element of $\mathcal{K}$, is congruent to the unit, the trivial knot $\circlearrowleft \in \mathbb{Q}_l[\mathcal{K}]$, modulo $\mathcal{K}_1$, for any knot can be untied by a finite times of changing crossings (consult for unknotting number, say, in [CDM]). Therefore the image of $h_l(\mathcal{K})$ ($\subset \widehat{\mathcal{K}}^l$) is contained in the subspace $\circlearrowleft + \mathcal{K}_1 \cdot \widehat{\mathbb{Q}_l[\mathcal{K}]}$. Hence the image of $\widehat{\mathcal{K}}^l$ should lie on the subspace. It is because the subspace is open in $\widehat{\mathbb{Q}_l[\mathcal{K}]}$, our map (2.8) is continuous as shown above and $h_l(\mathcal{K})$ is dense in $\widehat{\mathcal{K}}_l$ by Proposition 2.24.(3). All elements of the subspace are invertible because it is known that the quotient $\widehat{\mathbb{Q}_l[\mathcal{K}]}/\mathcal{K}_1$ is 1-dimensional and generated by $\circlearrowleft$. Thus the claim is obtained. □

The author is not sure if our above two maps are injective or not.

There is a natural map

$$i : \mathcal{K} \to \widehat{\mathbb{Q}[\mathcal{K}]} \ (\subset \widehat{R[\mathcal{K}]}) \qquad (R = \mathbb{C}, \mathbb{Q}_l) \tag{2.11}$$

induced from the natural inclusion

$$\mathcal{K} \hookrightarrow \mathbb{Q}[\mathcal{K}].$$

Remark 2.29. The Kontsevich invariant $Z : \mathcal{K} \to \widehat{CD}$ is a knot invariant which is a composition of i with the isomorphism $\widehat{\mathbb{C}[\mathcal{K}]} \simeq \widehat{CD}$ constructed in [K]. Here $\widehat{CD}$ stands for the $\mathbb{C}$-vector space (completed by degree) of chord diagrams. The invariant is conjectured to be perfect, i.e., the map Z is injective (cf. [O2] Conjecture 3.2). So the map i is also expected to be injective. It is direct to see that the map i factors through the maps (2.5) and (2.8). Therefore the map $h : \mathcal{K} \to \widehat{\mathcal{K}}$

is conjectured to be injective in Conjecture 2.11 since otherwise the Kontsevich invariant would fail to be perfect (cf. Remark 2.12)

2.3. **Absolute Galois action.** The group $G\widehat{\mathcal{K}}$ of profinite knots is introduced as the group of fraction of the topological monoid $\widehat{\mathcal{K}}$ in Definition 2.30 and its basic property is shown in Theorem 2.31. A continuous action of the profinite Grothendieck-Teichmüler group $\widehat{GT}$ (cf. Definition 1.6) on $G\widehat{\mathcal{K}}$ is established in Definition 2.34-Theorem 2.36. As a result of our construction, an action of the absolute Galois group $G_{\mathbb{Q}}$ of the rational number field on $G\widehat{\mathcal{K}}$ is obtained (Theorem 2.40). We post several projects and problems on this Galois representation in the end.

Definition 2.30. The *group of profinite knots* $G\widehat{\mathcal{K}}$ is defined to be the group of fraction of the monoid $\widehat{\mathcal{K}}$, i.e., the quotient space of $\widehat{\mathcal{K}}^2$ by the equivalent relations $(r,s) \approx (r',s')$ if $r\sharp s'\sharp t \sim r'\sharp s\sharp t$ for some profinite knot t, i.e. $r\sharp s'\sharp t = r'\sharp s\sharp t$ holds in $\widehat{\mathcal{K}}$. Occasionally we denote the equivalent class $[(r,s)]$ by $\frac{r}{s}$.

For $[(r_1,s_1)]$ and $[(r_2,s_2)] \in G\widehat{\mathcal{K}}$, define its product by

$$[(r_1,s_1)]\sharp[(r_2,s_2)] := [(r_1\sharp r_2, s_1\sharp s_2)] \in G\widehat{\mathcal{K}}, \qquad \text{i.e.} \qquad \frac{r_1}{s_1}\sharp\frac{r_2}{s_2} = \frac{r_1\sharp r_2}{s_1\sharp s_2} \in G\widehat{\mathcal{K}}.$$

We encode $G\widehat{\mathcal{K}}$ with the quotient topology of $\widehat{\mathcal{K}}^2$.

Theorem 2.31. *(1). The product $\sharp$ is well-defined on $G\widehat{\mathcal{K}}$. The set $G\widehat{\mathcal{K}}$ forms a topological commutative group.*

(2). It is a non-trivial group. Actually it is an infinite group.

Proof. (1). It is easy to see that $\sharp$ is well-defined and $G\widehat{\mathcal{K}}$ forms a commutative group with unit

$$e = (\circlearrowleft, \circlearrowleft)$$

by Theorem 2.14.

Consider the commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{K}}^2 \times \widehat{\mathcal{K}}^2 & \xrightarrow{\sharp} & \widehat{\mathcal{K}}^2 \\ \downarrow & & \downarrow \\ G\widehat{\mathcal{K}} \times G\widehat{\mathcal{K}} & \xrightarrow{\sharp} & G\widehat{\mathcal{K}}. \end{array}$$

Since the upper map is continuous by Theorem 2.14 and the surjection $\widehat{\mathcal{K}}^2 \to G\widehat{\mathcal{K}}$ is continuous by definition, it follows that the map $\sharp$ is continuous.

Let $\tau : \widehat{\mathcal{K}}^2 \to \widehat{\mathcal{K}}^2$ be the switch map sending $(r,s) \mapsto (s,r)$. It is easy to see that it is continuous and it induces the inverse map on $G\widehat{\mathcal{K}}$. Then by the commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{K}}^2 & \xrightarrow{\tau} & \widehat{\mathcal{K}}^2 \\ \downarrow & & \downarrow \\ G\widehat{\mathcal{K}} & \longrightarrow & G\widehat{\mathcal{K}}, \end{array}$$

the inverse map is also continuous.

(2). By Proposition 2.24 and 2.28, there is a continuous monoid homomorphism

$$\widehat{\mathcal{K}} \to \widehat{\mathcal{K}}^l \to \widehat{\mathbb{Q}_l[\mathcal{K}]} \qquad (2.12)$$

for a prime l. By Proposition 2.28, the image lies on $\widehat{\mathbb{Q}_l[\mathcal{K}]}^{\times}$. Whence it induces a continuous group homomorphism

$$G\widehat{\mathcal{K}} \to \widehat{\mathbb{Q}_l[\mathcal{K}]}^{\times}. \tag{2.13}$$

Thus it is enough to show that the image of the composition of the maps (2.12) and $h : \mathcal{K} \to \widehat{\mathcal{K}}$ is infinite set. The claim is obvious because this map is equal to i in (2.11) and the Kontsevich invariant takes infinite number of values (cf. Remark 2.29) □

A reason why we introduce $G\widehat{\mathcal{K}}$ is that we need to treat the inverse of Λ_f in Figure 2.24 when we let $\widehat{GT}$ act on profinite knots (cf. Definition 2.34).

We note that the natural morphism, which we call the *arithmetic realization map*,

$$h' : \mathcal{K} \to G\widehat{\mathcal{K}} \tag{2.14}$$

sending $K \mapsto [(K, \circlearrowleft)]$ is a homomorphism as monoid. By abuse of notations, we occasionally denote the image $h'(K)$ by the same symbol K. Related to Conjecture 2.11,

Problem 2.32. Is the arithmetic realization map h' injective?

On a structure of $G\widehat{\mathcal{K}}$, we pose

Problem 2.33. Is $G\widehat{\mathcal{K}}$ a profinite group?

By [RZ], to show that $G\widehat{\mathcal{K}}$ is a profinite group, we must show that it is compact, Hausdorff and totally-disconnected. The author is not aware of any one of their validities. It is worthy to note that the set $\widehat{\mathcal{T}}$ of isotopy classes of profinite tangles is not compact, hence not a profinite space. It is because the map $|\pi_0| : \widehat{\mathcal{T}} \to \mathbb{N}$ taking the number of connected components of each profinite tangles is continuous and surjective to the non-compact space $\mathbb{N}$.

Definition 2.34. Let (r, s) be a pair of profinite knots with $r = \gamma_{1,m} \cdots \gamma_{1,2} \cdot \gamma_{1,1}$ and $s = \gamma_{2,n} \cdots \gamma_{2,2} \cdot \gamma_{2,1}$ ($\gamma_{i,j}$: profinite fundamental tangle). For $\sigma = (\lambda, f) \in \widehat{GT}$ (hence $\lambda \in \widehat{\mathbb{Z}}^{\times}, f \in \widehat{F}_2$), define its action by

$$\sigma\left(\frac{r}{s}\right) := \frac{\sigma(r)}{\sigma(s)} := \frac{\{\sigma(\gamma_{1,m}) \cdots \sigma(\gamma_{1,2}) \cdot \sigma(\gamma_{1,1})\}\sharp(\Lambda_f)^{\sharp\alpha(s)}}{\{\sigma(\gamma_{2,n}) \cdots \sigma(\gamma_{2,2}) \cdot \sigma(\gamma_{2,1})\}\sharp(\Lambda_f)^{\sharp\alpha(r)}} \in G\widehat{\mathcal{K}}. \tag{2.15}$$

It is well-defined by Proposition 2.35 and Theorem 2.36. Here

$$\sigma(r) := \frac{\{\sigma(\gamma_{1,m}) \cdots \sigma(\gamma_{1,2}) \cdot \sigma(\gamma_{1,1})\}}{(\Lambda_f)^{\sharp\alpha(r)}} \text{ and } \sigma(s) := \frac{\{\sigma(\gamma_{2,n}) \cdots \sigma(\gamma_{2,2}) \cdot \sigma(\gamma_{2,1})\}}{(\Lambda_f)^{\sharp\alpha(s)}} \in G\widehat{\mathcal{K}}$$

are defined as follows:

(1) When $\gamma_{i,j} = a^{\epsilon}_{k,l}$, we define

$$\sigma(\gamma_{i,j}) := \gamma_{i,j} \cdot f^{s(\gamma_{i,j})}_{1\cdots k,k+1,k+2}$$

Here $f^{s(\gamma_{i,j})}_{1\cdots k,k+1,k+2} = ev_{1,\epsilon_1}(f^{\uparrow\epsilon_2\epsilon_3}) \otimes e^{\epsilon_4}_l$ with $s(\gamma_{i,j}) = \epsilon_1\epsilon_2\epsilon_3\epsilon_4 \in \{\uparrow,\downarrow\}^{k+l+2}$ ($\epsilon_1 \in \{\uparrow,\downarrow\}^k$, $\epsilon_2, \epsilon_3 \in \{\uparrow,\downarrow\}$, $\epsilon_4 \in \{\uparrow,\downarrow\}^l$). It is also described by $f^{s(\gamma_{i,j})}_{1\cdots k,k+1,k+2} = (f_{1\cdots k,k+1,k+2} \otimes e_l, s(\gamma_{i,j})) \in \widehat{B}$ with

$$f_{1\cdots k,k+1,k+2} \otimes e_l = f(x_{1\cdots k,k+1}, x_{k+1,k+2}) \in \widehat{B}_{k+l+2}$$

where $x_{1\cdots k,k+1}$ and $x_{k+1,k+2}$ are regarded as elements of $\widehat{B}_{k+l+2}$. We mean $f_{1\cdots k,k+1,k+2}\otimes e_l$ by the trivial braid $e_{l+2}\in\widehat{B}_{l+2}$ when $k=0$. Figure 2.22 depicts the action. Here the thickened black band stands for the trivial braid e_k with k-strings.

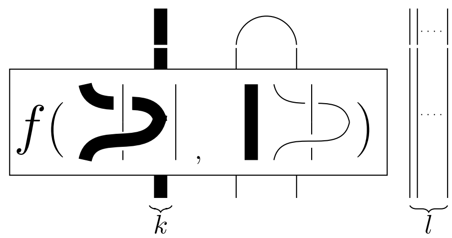


FIGURE 2.22. $\sigma(a^{\epsilon}_{k,l})$

(2) When $\gamma_{i,j}=b^{\epsilon}_n=(b_n,\epsilon)\in\widehat{B}$, we define

$$\sigma(\gamma_{i,j}):=(\sigma(b_n),\epsilon)$$

which is nothing but the image of $b_n\in\widehat{B}_n$ by the $\widehat{GT}$-action on $\widehat{B}_n$ explained in §1.

(3) When $\gamma_{i,j}=c^{\epsilon}_{k,l}$, we define

$$\sigma(\gamma_{i,j}):=f^{-1,t(\gamma_{i,j})}_{1\cdots k,k+1,k+2}\cdot\gamma_{i,j}$$

with $f^{-1,t(\gamma_{i,j})}_{1\cdots k,k+1,k+2}=\left(f^{-1}_{1\cdots k,k+1,k+2}\otimes e_l,t(\gamma_{i,j})\right)\in\widehat{B}$. Figure 2.23 depicts the action.

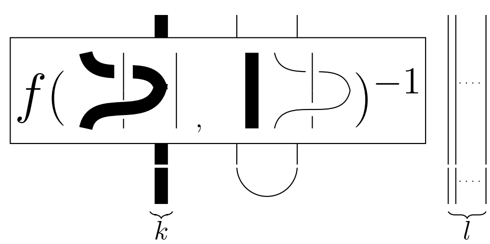


FIGURE 2.23. $\sigma(c^{\epsilon}_{k,l})$

The symbol Λ_f represents the profinite tangle

$$a^{\frown}_{0,0}\cdot a^{\frown\downarrow\uparrow}_{0,2}\cdot(e^{\downarrow}_1\otimes f)\cdot c^{\downarrow\smile\uparrow}_{1,1}\cdot c^{\smile}_{0,0}$$

(cf. Figure 2.24).

The symbol $\alpha(r)$ (resp. $\alpha(s)$) means the number of annihilations; the cardinality of the set $\{j|\gamma_{i,j}\in A\}$ for $i=1$ (resp. $i=2$) and $(\Lambda_f)^{\sharp\alpha(r)}$ (resp. $(\Lambda_f)^{\sharp\alpha(s)}$) means the $\alpha(r)$-th (resp. the $\alpha(s)$-th) power of Λ_f with respect to $\sharp$. Particularly we have

$$\sigma(\circlearrowleft)=\frac{\circlearrowleft}{\Lambda_f}\in G\widehat{\mathcal{K}} \tag{2.16}$$

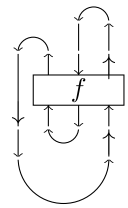


FIGURE 2.24. Λ_f

Proposition 2.35. *Let $\sigma = (\lambda, f) \in \widehat{GT}$.*

(1) *If $r = \gamma_{1,m} \cdots \gamma_{1,2} \cdot \gamma_{1,1}$ ($\gamma_{1,j}$: a profinite fundamental tangle) is a profinite knot, then $\{\sigma(\gamma_{1,m}) \cdots \sigma(\gamma_{1,2}) \cdot \sigma(\gamma_{1,1})\}$ is again a profinite knot.*

(2) *The profinite tangle Λ_f (Figure 2.24) is a profinite knot.*

Proof. (1). When $\gamma_{1,j} = b_n^\epsilon$, since the projection

$$p : \widehat{B}_n \to \mathfrak{S}_n$$

is $\widehat{GT}$-equivalent (the action on $\mathfrak{S}_n$ means the trivial action), the skeleton never change, i.e. $\mathbb{S}(\sigma(\gamma)) = \mathbb{S}(\gamma)$.

When $\gamma_{1,j} = a_{k,l}^\epsilon$ (resp. $c_{k,l}^\epsilon$), the skeleton $\mathbb{S}(\sigma(\gamma))$ is obtained by connecting $k + l + 2$ straight bars on the top (resp. bottom) of $\mathbb{S}(\gamma)$ because $f \in \widehat{F}_2 \subset \widehat{P}_3$.

Therefore $\{\sigma(\gamma_{1,m}) \cdots \sigma(\gamma_{1,2}) \cdot \sigma(\gamma_{1,1})\}$ is again a profinite knot.

(2). By Figure 2.24, it is easy because $f \in \widehat{F}_2 \subset \widehat{P}_3$. □

Theorem 2.36. *The equation* (2.15) *determines a well-defined $\widehat{GT}$-action on $G\widehat{\mathcal{K}}$. Namely,*

(1). $\sigma(\frac{r_1}{s_1}) = \sigma(\frac{r_2}{s_2}) \in G\widehat{\mathcal{K}}$ if $r_1 \sim r_2$ and $s_1 \sim s_2$, i.e. if $r_1 = r_2$ and $s_1 = s_2$ in $\widehat{\mathcal{K}}$.

(2). $\sigma(\frac{r_1}{s_1}) = \sigma(\frac{r_2}{s_2}) \in G\widehat{\mathcal{K}}$ if $(r_1, s_1) \approx (r_2, s_2)$, i.e. if $\frac{r_1}{s_1} = \frac{r_2}{s_2}$ in $G\widehat{\mathcal{K}}$.

(3). $\sigma_1(\sigma_2(x)) = (\sigma_1 \circ \sigma_2)(x)$ for any $\sigma_1, \sigma_2 \in \widehat{GT}$ and $x \in G\widehat{\mathcal{K}}$.

Furthermore $G\widehat{\mathcal{K}}$ forms a topological $\widehat{GT}$-module. Namely,

(4). the action is compatible with the group structure, i.e.

$$\sigma(e) = e, \qquad \sigma(x \sharp y) = \sigma(x) \sharp \sigma(y), \qquad \sigma(x^{-1}) = \sigma(x)^{-1}$$

for any $\sigma \in \widehat{GT}$ and $x, y \in G\widehat{\mathcal{K}}$.

(5). the action is continuous.

Proof. (1). Firstly we prove that $\sigma((r_1, s)) = \sigma((r_2, s)) \in \widehat{\mathcal{K}}^2$ when r_1 is isotopic to r_2 for $\sigma = (\lambda, f) \in \widehat{GT}$. We may further assume that r_1 is obtained from r_2 by a single operation of one of the moves (T1)-(T6).

- If it is (T1), it is clear.
- If it is (T2), is is immediate because $\sigma(b_2) \cdot \sigma(b_1) = \sigma(b_2 b_1)$ holds for $b_1, b_2 \in \widehat{B}_n$.
- If it is (T3), we may further assume that its T_1 and T_2 in (T3) are both fundamental profinite tangles. Then by Proposition 1.16 and Proposition

2.37

$$\sigma(e_{n_1}^{t(T_1)}\otimes T_2)\cdot\sigma(T_1\otimes e_{m_2}^{s(T_2)})$$
$$=f_{[n_1],[n_2],[0]}^{-1,(t(T_1),t(T_2))}\cdot(e_{n_1}^{t(T_1)}\otimes\sigma(T_2))\cdot f_{[n_1],[m_2],[0]}^{(t(T_1),s(T_2))}\cdot(\sigma(T_1)\otimes e_{m_2}^{s(T_2)}),$$

by (T4)

$$=f_{[n_1],[n_2],[0]}^{-1,(t(T_1),t(T_2))}\cdot(e_{n_1}^{t(T_1)}\otimes\sigma(T_2))\cdot(\sigma(T_1)\otimes e_{m_2}^{s(T_2)})\cdot f_{[m_1],[m_2],[0]}^{(s(T_1),s(T_2))}$$
$$=f_{[n_1],[n_2],[0]}^{-1,(t(T_1),t(T_2))}\cdot(\sigma(T_1)\otimes e_{n_2}^{t(T_2)})\cdot(e_{m_1}^{s(T_1)}\otimes\sigma(T_2))\cdot f_{[m_1],[m_2],[0]}^{(s(T_1),s(T_2))}$$
$$=(\sigma(T_1)\otimes e_{n_2}^{t(T_2)})\cdot f_{[m_1],[n_2],[0]}^{-1,(s(T_1),t(T_2))}\cdot(e_{m_1}^{s(T_1)}\otimes\sigma(T_2))\cdot f_{[m_1],[m_2],[0]}^{(s(T_1),s(T_2))}$$
$$=\sigma(T_1\otimes e_{n_2}^{t(T_2)})\cdot\sigma(e_{m_1}^{s(T_1)}\otimes T_2).$$

Whence (T3) is preserved by the GT-action.

- If it is (T4), we may assume that T in (T4) is a fundamental profinite tangle. Then by Proposition 1.18 and Proposition 2.37

$$\sigma(ev_{k,t(T)}(b_l^\epsilon))\cdot\sigma(e_{k-1}^{s_1}\otimes T\otimes e_{l-k}^{s_2})$$
$$=f_{[k'-1],[n],[l-k']}^{-1,(t_1,t(T),t_2)}\cdot ev_{k,t(T)}(\sigma(b_l^\epsilon))\cdot f_{[k-1],[n],[l-k]}^{(s_1,t(T),s_2)}$$
$$\cdot f_{[k-1],[n],[l-k]}^{-1,(s_1,t(T),s_2)}\cdot(e_{k-1}^{s_1}\otimes\sigma(T)\otimes e_{l-k}^{s_2})\cdot f_{[k-1],[m],[l-k]}^{(s_1,s(T),s_2)}$$
$$=f_{[k'-1],[n],[l-k']}^{-1,(t_1,t(T),t_2)}\cdot ev_{k,t(T)}(\sigma(b_l^\epsilon))\cdot(e_{k-1}^{s_1}\otimes\sigma(T)\otimes e_{l-k}^{s_2})\cdot f_{[k-1],[m],[l-k]}^{(s_1,s(T),s_2)},$$

by (T4)

$$=f_{[k'-1],[n],[l-k']}^{-1,(t_1,t(T),t_2)}\cdot(e_{k'-1}^{t_1}\otimes\sigma(T)\otimes e_{l-k'}^{t_2})\cdot ev^{k',s(T)}(\sigma(b_l^\epsilon))\cdot f_{[k-1],[m],[l-k]}^{(s_1,s(T),s_2)},$$
$$=f_{[k'-1],[n],[l-k']}^{-1,(t_1,t(T),t_2)}\cdot(e_{k'-1}^{t_1}\otimes\sigma(T)\otimes e_{l-k'}^{t_2})\cdot f_{[k'-1],[m],[l-k']}^{(t_1,s(T),t_2)}$$
$$\cdot f_{[k'-1],[m],[l-k']}^{-1,(t_1,s(T),t_2)}\cdot ev^{k',s(T)}(\sigma(b_l^\epsilon))\cdot f_{[k-1],[m],[l-k]}^{(s_1,s(T),s_2)}$$
$$=\sigma(e_{k'-1}^{t_1}\otimes T\otimes e_{l-k'}^{t_2})\cdot\sigma(ev^{k',s(T)}(b_l^\epsilon)).$$

Whence (T4) is preserved by the action.

- If it is (T5), we have

$$\sigma(a_{k+1,l-1}^{\epsilon'})\cdot\sigma(c_{k,l}^\epsilon)=a_{k+1,l-1}^{\epsilon'}\cdot f_{1\cdots k+1,k+2,k+3}^{s(c_{k,l}^\epsilon)}\cdot f_{1\cdots k,k+1,k+2}^{-1,s(c_{k,l}^\epsilon)}\cdot c_{k,l}^\epsilon.$$

By the pentagon equation (1.4)

$$=a_{k+1,l-1}^{\epsilon'}\cdot f_{1\cdots k,k+1,k+2\ \ k+3}^{-1,s(c_{k,l}^\epsilon)}\cdot f_{k+1,k+2,k+3}^{s(c_{k,l}^\epsilon)}\cdot f_{1\cdots k,k+1\ \ k+2,k+3}^{s(c_{k,l}^\epsilon)}\cdot c_{k,l}^\epsilon,$$

by (T4) and Lemma 1.8

$$=a_{k+1,l-1}^{\epsilon'}\cdot f_{k+1,k+2,k+3}^{s(c_{k,l}^\epsilon)}\cdot c_{k,l}^\epsilon.$$

It looks that (T5) is not preserved by the GT-action. But actually it means that $\sigma(r_1)$ is obtained by an insertion of $f_{k+1,k+2,k+3}^{s(c_{k,l}^\epsilon)}$ between $a_{k+1,l-1}^{\epsilon'}$ and $c_{k,l}^\epsilon$ in $\sigma(r_2)$. Thus $\sigma(r_1)=\sigma(r_2)\sharp\Lambda_f$. Because $\alpha(r_1)=\alpha(r_2)+1$, we may say that (T5) is compatible with the action by (2.15). The second equality can be proved in the same way.

- If it is (T6), again by Proposition 1.16 and Proposition 2.37,

$$\begin{aligned}&\sigma(a^{\epsilon}_{k,l})\cdot\sigma((\sigma^{\epsilon'}_{k+1})^{2c+1})\\ &\quad=a^{\epsilon}_{k,l}\cdot f^{s(a^{\epsilon}_{k,l})}_{1\cdots k,k+1,k+2}\cdot f^{-1,s(a^{\epsilon}_{k,l})}_{1\cdots k,k+1,k+2}\cdot(\sigma^{\epsilon'}_{k+1})^{\lambda(2c+1)}\cdot f^{s(a^{\bar{\epsilon}}_{k,l})}_{1\cdots k,k+1,k+2}\\ &\quad=a^{\epsilon}_{k,l}\cdot(\sigma^{\epsilon'}_{k+1})^{\lambda(2c+1)}\cdot f^{s(a^{\bar{\epsilon}}_{k,l})}_{1\cdots k,k+1,k+2},\end{aligned}$$

by $\lambda\equiv 1 \pmod 2$ and (T6)

$$=a^{\bar{\epsilon}}_{k,l}\cdot f^{s(a^{\bar{\epsilon}}_{k,l})}_{1\cdots k,k+1,k+2}=\sigma(a^{\bar{\epsilon}}_{k,l}).$$

The case for $c^{\epsilon}_{k,l}$ can be checked in the same way.

Secondly we prove that $\sigma((r,s_1))=\sigma((r,s_2))\in\widehat{\mathcal{K}}^2$ when s_1 is isotopic to s_2. But it can be proved in a similar way to the above. Hence our claim of (1) is obtained.

(2). By definition,

$$r_1\sharp s_2\sharp t=r_2\sharp s_1\sharp t$$

in $\mathcal{K}$ for some profinite knot t. By the definition of $\sharp$,

$$\sigma(k_1\sharp k_2)\sharp\sigma(\circlearrowleft)=\sigma(k_1)\sharp\sigma(k_2)$$

equivalently

$$\sigma(k_1\sharp k_2)=\sigma(k_1)\sharp\sigma(k_2)\sharp\Lambda_f, \tag{2.17}$$

holds in $G\widehat{\mathcal{K}}$ for any profinite knot k_1 and k_2. Therefore our claim is immediate because

$$\sigma(r_1\sharp s_2\sharp t)=\sigma(r_1)\sharp\sigma(s_2)\sharp\sigma(t)\sharp\Lambda_f\sharp\Lambda_f$$

and

$$\sigma(r_2\sharp s_1\sharp t)=\sigma(r_2)\sharp\sigma(s_1)\sharp\sigma(t)\sharp\Lambda_f\sharp\Lambda_f.$$

(3). For $\sigma_1=(\lambda_1,f_1)$ and $\sigma_2=(\lambda_2,f_2)\in\widehat{GT}$, put $\sigma_3=\sigma_2\circ\sigma_1\in\widehat{GT}$. Hence, by (1.5), $\sigma_3=(\lambda_3,f_3)$ with $\lambda_3=\lambda_2\lambda_1$ and

$$f_3=f_2\cdot\sigma_2(f_1)=f_2\cdot f_1(x^{\lambda_2},f_2^{-1}y^{\lambda_2}f_2)(=:f_2\circ f_1). \tag{2.18}$$

Firstly we note that

$$\sigma_3(\gamma_{i,j})=\sigma_2\left(\sigma_1(\gamma_{i,j})\right).$$

When $\gamma_{i,j}=a^{\epsilon}_{k,l}$ or $c^{\epsilon}_{k,l}$, the equality is derived from (2.18). When $\gamma_{i,j}=b^{\epsilon}_n$, it is easy because of the $\widehat{GT}$-action on $\widehat{B}_n$.

Secondly by definition we have

$$\sigma_2(\Lambda_{f_1})=\{\sigma_2(a_{0,0})\cdot\sigma_2(a_{0,2})\cdot\sigma_2(e_1\otimes f_1)\cdot\sigma_2(c_{1,1})\cdot\sigma_2(c_{0,0})\}/\Lambda^{\sharp 2}_{f_2}.$$

By Proposition 1.16 and Proposition 2.37

$$\begin{aligned}&=\{a_{0,0}\cdot a_{0,2}\cdot(f_2^{-1})_{1,2,3}\cdot(f_2^{-1})_{1,23,4}\cdot(e_1\otimes\sigma_2(f_1))\\ &\qquad\cdot(f_2)_{1,23,4}\cdot(f_2)_{1,2,3}\cdot(f_2^{-1})_{1,2,3}\cdot c_{1,1}\cdot c_{0,0}\}/\Lambda^{\sharp 2}_{f_2}\\ &=\{a_{0,0}\cdot a_{0,2}\cdot(f_2^{-1})_{1,2,3}\cdot(f_2^{-1})_{1,23,4}\cdot(e_1\otimes\sigma_2(f_1))\cdot(f_2)_{1,23,4}\cdot c_{1,1}\cdot c_{0,0}\}/\Lambda^{\sharp 2}_{f_2},\end{aligned}$$

by the pentagon equation (1.4)

$$
\begin{aligned}
&= \{a_{0,0} \cdot a_{0,2} \cdot (f_2^{-1})_{12,3,4} \cdot (f_2^{-1})_{1,2,34} \cdot (f_2)_{2,3,4} \cdot (e_1 \otimes \sigma_2(f_1)) \\
&\qquad \cdot (f_2)_{1,23,4} \cdot c_{1,1} \cdot c_{0,0}\}/\Lambda_{f_2}^{\sharp 2},
\end{aligned}
$$

by a successive application of (T6) and Lemma 1.8

$$
\begin{aligned}
&= \{a_{0,0} \cdot a_{0,2} \cdot (f_2)_{2,3,4} \cdot (e_1 \otimes \sigma_2(f_1)) \cdot c_{1,1} \cdot c_{0,0}\}/\Lambda_{f_2}^{\sharp 2} \\
&= \{a_{0,0} \cdot a_{0,2} \cdot (e_1 \otimes f_2 \cdot \sigma_2(f_1)) \cdot c_{1,1} \cdot c_{0,0}\}/\Lambda_{f_2}^{\sharp 2} \\
&= \Lambda_{f_2 \circ f_1} \sharp \sigma_2(\circlearrowleft)^{\sharp 2}.
\end{aligned}
$$

We note that in the above computation we omit the symbol ϵ of orientation.

Finally

$$
\begin{aligned}
\sigma_2\left(\sigma_1\left(\frac{r}{s}\right)\right) &= \sigma_2\left(\frac{\{\sigma_1(\gamma_{1,m})\cdots\sigma_1(\gamma_{1,2})\cdot\sigma_1(\gamma_{1,1})\}\sharp(\Lambda_{f_1})^{\sharp\alpha(s)}}{\{\sigma_1(\gamma_{2,n})\cdots\sigma_1(\gamma_{2,2})\cdot\sigma_1(\gamma_{2,1})\}\sharp(\Lambda_{f_1})^{\sharp\alpha(r)}}\right) \\
&= \frac{\sigma_2(\{\sigma_1(\gamma_{1,m})\cdots\sigma_1(\gamma_{1,2})\cdot\sigma_1(\gamma_{1,1})\})\sharp\sigma_2(\Lambda_{f_1})^{\sharp\alpha(s)}\sharp\Lambda_{f_2}^{\sharp\alpha(s)}}{\sigma_2(\{\sigma_1(\gamma_{2,n})\cdots\sigma_1(\gamma_{2,2})\cdot\sigma_2(\gamma_{2,1})\})\sharp\sigma_2(\Lambda_{f_1})^{\sharp\alpha(r)}\sharp\Lambda_{f_2}^{\sharp\alpha(r)}} \\
&= \frac{\{\sigma_2(\sigma_1(\gamma_{1,m}))\cdot\cdots\sigma_2(\sigma_1(\gamma_{1,1}))\}\sharp\Lambda_{f_2}^{\sharp\alpha(s)}\sharp(\Lambda_{f_3})^{\sharp\alpha(s)}\sharp\sigma_2(\circlearrowleft)^{\sharp 2\alpha(s)}\sharp\Lambda_{f_2}^{\sharp\alpha(s)}}{\{\sigma_2(\sigma_1(\gamma_{2,n}))\cdot\cdots\sigma_2(\sigma_1(\gamma_{2,1}))\}\sharp\Lambda_{f_2}^{\sharp\alpha(r)}\sharp(\Lambda_{f_3})^{\sharp\alpha(r)}\sharp\sigma_2(\circlearrowleft)^{\sharp 2\alpha(r)}\sharp\Lambda_{f_2}^{\sharp\alpha(r)}} \\
&= \frac{\{\sigma_3(\gamma_{1,m})\cdots\sigma_3(\gamma_{1,2})\cdot\sigma_3(\gamma_{1,1})\}\sharp(\Lambda_{f_3})^{\sharp\alpha(s)}}{\{\sigma_3(\gamma_{2,n})\cdots\sigma_3(\gamma_{2,2})\cdot\sigma_3(\gamma_{2,1})\}\sharp(\Lambda_{f_3})^{\sharp\alpha(r)}} = \sigma_3\left(\frac{r}{s}\right)
\end{aligned}
$$

by (2.16) and (2.17).

(4). Let $x = r_1/s_1$ and $y = r_2/s_2$ with profinite knots r_1, r_2, s_1, s_2. Then by (2.17) it is easy to see

$$
\begin{aligned}
\sigma(x\sharp y) = \sigma(\frac{r_1\sharp r_2}{s_1\sharp s_2}) &= \frac{\sigma(r_1\sharp r_2)}{\sigma(s_1\sharp s_2)} = \frac{\sigma(r_1)\sharp\sigma(r_2)\sharp\Lambda_f}{\sigma(s_1)\sharp\sigma(s_2)\sharp\Lambda_f} = \frac{\sigma(r_1)\sharp\sigma(r_2)}{\sigma(s_1)\sharp\sigma(s_2)} \\
&= \frac{\sigma(r_1)}{\sigma(s_1)}\sharp\frac{\sigma(r_2)}{\sigma(s_2)} = \sigma(\frac{r_1}{s_1})\sharp\sigma(\frac{r_2}{s_2}) = \sigma(x)\sharp\sigma(y).
\end{aligned}
$$

The inverse is also easy to check.

(5). We recall that $\widehat{\mathcal{K}'}^{\mathrm{seq}}$ (cf. the proof of Theorem 2.14. (2)) is the set of finite consistent sequences of profinite fundamental tangles $\gamma_n \cdots \gamma_2 \cdot \gamma_1$ with a single connected component and with $(\gamma_n, \gamma_1) = (\curvearrowleft, \curvearrowright)$. We define the map

$$
A : \widehat{GT} \times \widehat{\mathcal{K}'}^{\mathrm{seq}} \times \widehat{\mathcal{K}'}^{\mathrm{seq}} \to \widehat{\mathcal{K}'}^{\mathrm{seq}} \times \widehat{\mathcal{K}'}^{\mathrm{seq}}
$$

by

$$
A(\sigma, r, s) = \Big(\{\sigma(\gamma_{1,m})\cdots\sigma(\gamma_{1,2})\cdot\sigma(\gamma_{1,1})\}\sharp(\Lambda_f)^{\sharp\alpha(s)}, \{\sigma(\gamma_{2,n})\cdots\sigma(\gamma_{2,2})\cdot\sigma(\gamma_{2,1})\}\sharp(\Lambda_f)^{\sharp\alpha(r)}\Big)
$$

for $\sigma = (\lambda, f)$, $r = \gamma_{1,m} \cdots \gamma_{1,2} \cdot \gamma_{1,1}$ and $s = \gamma_{2,n} \cdots \gamma_{2,2} \cdot \gamma_{2,1}$ ($\gamma_{i,j}$: profinite fundamental tangle). We know that the $\widehat{GT}$-action on $\widehat{B}_n$ and the map $\widehat{GT} \to \widehat{B}_3$: $\sigma = (\lambda, f) \mapsto f$ are continuous, so the map A is continuous. Since the diagram

below is commutative

$$\begin{array}{ccc} \widehat{GT} \times \widehat{\mathcal{K}'}^{\mathrm{seq}} \times \widehat{\mathcal{K}'}^{\mathrm{seq}} & \xrightarrow{A} & \widehat{\mathcal{K}'}^{\mathrm{seq}} \times \widehat{\mathcal{K}'}^{\mathrm{seq}} \\ \downarrow & & \downarrow \\ \widehat{GT} \times G\widehat{\mathcal{K}} & \longrightarrow & G\widehat{\mathcal{K}} \end{array}$$

and the projection $\widehat{\mathcal{K}'}^{\mathrm{seq}} \times \widehat{\mathcal{K}'}^{\mathrm{seq}} \twoheadrightarrow G\widehat{\mathcal{K}}$ is continuous, the lower map is also continuous. $\square$

The following is required to prove Theorem 2.36.

Proposition 2.37. *Let $k, l, m_1, m_2 \geqslant 0$ and $\epsilon_i \in \{\uparrow,\downarrow\}^{m_i}$ $(i = 1, 2)$. For any $\sigma \in \widehat{GT}$,*

$$\sigma(a^{\epsilon_1\epsilon\epsilon_2}_{m_1+k,l+m_2}) = f^{-1,\epsilon_t}_{[m_1],[k+l],[m_2]} \cdot \left(e^{\epsilon_1}_{m_1} \otimes \sigma(a^{\epsilon}_{k,l}) \otimes e^{\epsilon_2}_{m_2}\right) \cdot f^{\epsilon_s}_{[m_1],[k+l+2],[m_2]}$$

with $\epsilon_t = t(a^{\epsilon_1\epsilon\epsilon_2}_{m_1+k,l+m_2})$ and $\epsilon_s = s(a^{\epsilon_1\epsilon\epsilon_2}_{m_1+k,l+m_2})$. And

$$\sigma(c^{\epsilon_1\epsilon\epsilon_2}_{m_1+k,l+m_2}) = f^{-1,\epsilon_t}_{[m_1],[k+l+2],[m_2]} \cdot \left(e^{\epsilon_1}_{m_1} \otimes \sigma(c^{\epsilon}_{k,l}) \otimes e^{\epsilon_2}_{m_2}\right) f^{\epsilon_s}_{[m_1],[k+l],[m_2]}$$

with $\epsilon_t = t(a^{\epsilon_1\epsilon\epsilon_2}_{m_1+k,l+m_2})$ and $\epsilon_s = s(a^{\epsilon_1\epsilon\epsilon_2}_{m_1+k,l+m_2})$.

Here $f^{\epsilon}_{[m_1],[M],[m_2]} \in \widehat{B}$ means $(f_{[m_1],[M],[m_2]}, \epsilon) \in \widehat{B}_{m_1+M+m_2} \times \{\uparrow,\downarrow\}^{m_1+M+m_2}$ with (see also (1.13))

$$\begin{aligned} f_{[m_1],[M],[m_2]} :=& f_{1\cdots m_1,m_1+1\cdots m_1+M-1,m_1+M} \cdot f_{1\cdots m_1,m_1+1\cdots m_1+M-2,m_1+M-1} \cdot \\ & \cdots f_{1\cdots m_1,m_1+1,m_1+2} \in \widehat{B}_{m_1+M+m_2}. \end{aligned}$$

Proof. We prove the first equality. To avoid the complexity, we again omit the symbol of orientations. By Definition 2.34.(1),

$$\begin{aligned} (2.19)\quad & f^{-1}_{[m_1],[k+l],[m_2]} \cdot (e_{m_1} \otimes \sigma(a_{k,l}) \otimes e_{m_2}) \cdot f_{[m_1],[k+l+2],[m_2]} \\ &= f^{-1}_{[m_1],[k+l],[m_2]} \cdot a_{m_1+k,l+m_2} \cdot f_{m_1+1\cdots m_1+k,m_1+k+1,m_1+k+2} \cdot f_{[m_1],[k+l+2],[m_2]}. \end{aligned}$$

- When $M \geqslant k+3$, by (T4), $f_{m_1+1\cdots m_1+k,m_1+k+1,m_1+k+2}$ commutes with $f_{1\cdots m_1,m_1+1\cdots m_1+M-1,m_1+M}$ and

$$a_{m_1+k,l+m_2} \cdot f_{1\cdots m_1,m_1+1\cdots m_1+M-1,m_1+M} = f_{1\cdots m_1,m_1+1\cdots m_1+M-3,m_1+M-2} \cdot a_{m_1+k,l+m_2}.$$

  Therefore

$$(2.19) = f^{-1}_{[m_1],[k],[l+m_2]} \cdot a_{m_1+k,l+m_2} \cdot f_{m_1+1\cdots m_1+k,m_1+k+1,m_1+k+2} \cdot f_{[m_1],[k+2],[l+m_2]}.$$

- When $M = k+1, k+2$, our calculation goes as follows.

$$\begin{aligned} (2.19) =& f^{-1}_{[m_1],[k],[l+m_2]} \cdot a_{m_1+k,l+m_2} \cdot f_{m_1+1\cdots m_1+k,m_1+k+1,m_1+k+2} \cdot \\ & f_{1\cdots m_1,m_1+1\cdots m_1+k+1,m_1+k+2} \cdot f_{1\cdots m_1,m_1+1\cdots m_1+k,m_1+k+1} \cdot f_{[m_1],[k],[l+m_2+2]}, \end{aligned}$$

  by the pentagon equation (1.4),

$$\begin{aligned} =& f^{-1}_{[m_1],[k],[l+m_2]} \cdot a_{m_1+k,l+m_2} \cdot f_{1\cdots m_1,m_1+1\cdots m_1+k,m_1+k+1\ m_1+k+2} \cdot \\ & f_{1\cdots m_1+k,m_1+k+1,m_1+k+2} \cdot f_{[m_1],[k],[l+m_2+2]}, \end{aligned}$$

by (T4) and Lemma 1.8,

$$= f^{-1}_{[m_1],[k],[l+m_2]} \cdot a_{m_1+k,l+m_2} \cdot f_{1\cdots m_1+k,m_1+k+1,m_1+k+2} \cdot f_{[m_1],[k],[l+m_2+2]}.$$

- When $M \leqslant k$, by (T4) again,
  $f_{1\cdots m_1+k,m_1+k+1,m_1+k+2}$ commutes with $f_{1\cdots m_1,m_1+1\cdots m_1+M-1,m_1+M}$ and

$$a_{m_1+k,l+m_2} \cdot f_{1\cdots m_1,m_1+1\cdots m_1+M-1,m_1+M} = f_{1\cdots m_1,m_1+1\cdots m_1+M-1,m_1+M} \cdot a_{m_1+k,l+m_2}.$$

Therefore

$$\begin{aligned}(2.19) &= f^{-1}_{[m_1],[k],[l+m_2]} \cdot a_{m_1+k,l+m_2} \cdot f_{[m_1],[k],[l+m_2+2]} \cdot f_{1\cdots m_1+k,m_1+k+1,m_1+k+2}, \\ &= a_{m_1+k,l+m_2} \cdot f_{1\cdots m_1+k,m_1+k+1,m_1+k+2} = \sigma(a_{m_1+k,l+m_2}).\end{aligned}$$

Hence we get the equality.

The second equality can be proved in the same way. □

Thus by Theorem 2.36, the $\widehat{GT}$-action

$$\widehat{GT} \to AutG\widehat{\mathcal{K}} \tag{2.20}$$

is established.

Remark 2.38. In [KT], it is explained that the category $\widehat{\mathcal{T}}(R)$ (R: a commutative ring containing $\mathbb{Q}$) of 'completed R-linear framed tangles' forms an R-linear ribbon category. From which they deduced an action of the proalgebraic Grothendieck-Teichmüller group $GT(R)$ [Dr2] on the space of 'completed R-linear knots' by a categorical arguments. An analogous categorical deduction of our Theorem 2.36 might be expectable. However a completely same argument does not seem to work. We may have a ribbon category $\widehat{\mathcal{T}}$ of profinite (framed) tangles but a difficulty here is that our $(\Lambda_f)^{-1}$ is not seemingly meaningful in $\widehat{\mathcal{T}}$, unlike the case of $\widehat{\mathcal{T}}(R)$. A technical care to remedy this might be required.

Remark 2.39. The Kontsevich knot invariant [Ko] is obtained by integrating a formal analogue of the KZ (Knizhnik-Zamolodchikov) equation. Bar-Natan [Ba], Kassel-Turaev [KT], Le-Murakami [LM] and Pieunikhin [P] gave a combinatorial reconstruction of the invariant by using an associator [Dr2]. An *associator* means a pair (μ, φ) with $\mu \in R^\times$ and an R-coefficient non-commutative formal power series φ with two variables satisfying specific relations which are analogues of our pentagon and hexagon equations (1.4)-(1.3) ([Dr2], see also [F1]). One of striking results [12] in Le-Murakami [LM] is the rationality of the Kontsevich invariant which follows from that the resulting invariant is, in fact, independent of φ (but depends on μ). Their result may suggest that the above $\widehat{GT}$-action (2.20) would depend only on $\lambda \in \widehat{\mathbb{Z}}^\times$ of $(\lambda, f) \in \widehat{GT}$, namely, the action would factor through $\widehat{\mathbb{Z}}^\times$. We remind that their proof of the independency is based on certain linear algebraic arguments, actually a vanishing of a (Harrison) cohomology of a chain complex associated with chords. But here we are working not on their proalgebraic setting but on the profinite setting where such vanishing result has not been established. And we do not know whether such a factorization would hold in our setting or not.

Finally we obtain a Galois representation on knots as an important consequence of Theorem 2.36.

[12] It might be amazing to know that Drinfel'd indicated it in [Dr1].

Theorem 2.40. *Fix an embedding from $\overline{\mathbb{Q}}$ in to $\mathbb{C}$. The group $G\widehat{\mathcal{K}}$ of profinite knots admits a non-trivial topological $G_{\mathbb{Q}}$-module structure. Namely there is a non-trivial continuous Galois representation*

$$\rho_0 : G_{\mathbb{Q}} \to AutG\widehat{\mathcal{K}}. \tag{2.21}$$

Proof. By Theorem 2.36, it is straightforward because in Theorem 1.9 we see that the absolute Galois group $G_{\mathbb{Q}}$ is mapped to $\widehat{GT}$. It is proved that $G\widehat{\mathcal{K}}$ is nontrivial in Theorem 2.31. The non-triviality of ρ_0 is a consequence of the example below because generally we have $K \neq \overline{K}$ in $G\widehat{\mathcal{K}}$: For instance, the left trefoil (the knot in the left below of Figure 2.14) and the right trefoil (its mirror image) are mapped to different elements by the map (2.13) because they are known to be separated by the Kontsevich invariant. (cf. Remark 2.29.) $\square$

Example 2.41. Especially when $\sigma \in G_{\mathbb{Q}}$ is equal to the complex conjugation morphism ς_0, it corresponds to $(\lambda, f) = (-1, 1) \in \widehat{GT}$ whose action on $\widehat{B}_n$ is given by $\sigma_i \mapsto \sigma_i^{-1}$ $(1 \leqslant i \leqslant n-1)$ (cf. Example 1.10). Whence the action of ς_0 on $G\widehat{\mathcal{K}}$ is particularly described by

$$\rho_0(\varsigma_0)\,(K) = \overline{K}$$

for $K \in \mathcal{K}$ because $\Lambda_1 = \circlearrowleft$. Here we denote the image of the arithmetic realization map h' (2.14) on an oriented knot K by the same symbol K and we mean the mirror image of the knot K by $\overline{K}$.

The easiest example is that the right trefoil knot is mapped to the left trefoil knot by the complex conjugation.

Project 2.42. In §1, the actions of $\widehat{GT}$ and $G_{\mathbb{Q}}$ on the profinite braid group $\widehat{B}_n$ are discussed. In Remark 1.14 it is explained in the language of algebraic geometry that the $G_{\mathbb{Q}}$-action on $\widehat{B}_n$ is caused by the homotopy exact sequence of the scheme-theoretic fundamental group of the quotient variety $Conf^n_{\mathfrak{S}_n}$ of the configuration space $Conf^n$. Whilst as for our Galois action on knots in Theorem 2.40, the author is not sure whether there is such a purely 'algebraic-geometrical' interpretation without usage of $\widehat{GT}$ or not. It should be our future research.

Asking the validity of an analogue of Belyĭ's theorem [Be] in (1.8) is particularly significant.

Problem 2.43. Is the absolute Galois action (2.21) on profinite knots faithful? If not, then what is the corresponding kernel field? And what is the arithmetic meaning of this?

This is also related to the problem discussed in Remark 2.39 above. In [F2], it will be explained that the corresponding kernel field is bigger than the maximal abelian extension $\mathbb{Q}(\mu_\infty)$ of $\mathbb{Q}$. Asking questions for each given knot is also worthy to discuss.

Problem 2.44. What is the stabilizer of each given (usual) knot? And what is the corresponding field?

Example 2.41 tells us that the stabilizer of amphicheiral knot, say, the figure eight knot (the knot in the left above of Figure 2.14), contains the subgroup $\{id, \varsigma_0\}$ of order 2 at least.

Detecting each Galois orbit of a given knot is our another problem. We only know that its mirror image lies on the same Galois orbit of a given usual knot. The problem formulated below is a first step towards it.

Problem 2.45. Let K_1 and K_2 be two usual knots. Let $\mathcal{N}$ be an open subgroup of $G\widehat{\mathcal{K}}$. When do two cosets $K_1 \cdot \mathcal{N}$ and $K_2 \cdot \mathcal{N}$ lie on the same Galois orbit? If so, then which $\sigma \in G_{\mathbb{Q}}$ connect them?

In Morishita [Mo], he observes a mysterious analogy between Gauss' quadratic reciprocity law in number theory and the duality of Gauss' linking numbers in 3-dimensional topology. It would be wonderful if the above problem could lead us to realize a direct relationship between them.

Project 2.46. There are various notions of equivalences for (framed) knots (and links) such as the Kirby moves (the Fenn-Rourke moves), the knot cobordism, the knot concordance, etc. Extending these notions into those for our profinite links and examining their behaviors under our Galois action is worthy to pursue. The following particularly discusses on the Kirby moves:

It is known (consult the standard textbook such as [O1]) that every closed connected oriented 3-manifold (three dimensional manifold) can be obtained by Dehn surgery on the three dimensional sphere S^3 along a certain framed link and two framed links result the same 3-manifold if and only if they are related by a finite number of the Kirby moves $\sim_{\mathrm{K}}$, generated by (K1) and (K2) in Figure 2.25 and 2.26. (Here the Figure 2.25 represents adding (resp. deleting) a unknot with

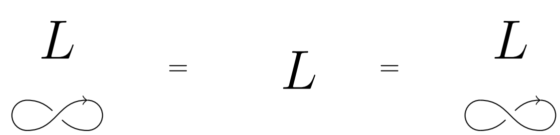


FIGURE 2.25. Kirby move (K1)

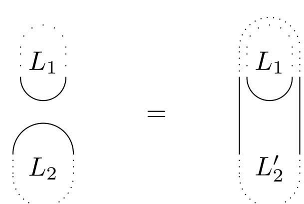


FIGURE 2.26. Kirby move (K2)

framing ± 1 to (resp. from) a framed link L. In the Figure 2.26, two framed links L_1 and L_2 are unlinked and L_2' is obtained by letting the black arc of L_2 encircling around the dotted arc of L_1.) Thus there is a surjection

$$DS: \mathcal{L}^{\mathrm{fr}} \twoheadrightarrow \mathbb{M}^3$$

from the set $\mathcal{L}^{\mathrm{fr}}$ of isotopy classes of framed oriented links to the set $\mathbb{M}^3$ of homeomorphism classes of closed oriented 3-manifolds. And $DS(L_1) = DS(L_2)$ if and only if $L_1 \sim_{\mathrm{K}} L_2$.

Our results in Theorem 2.36 can be extended into profinite framed links in an appropriate way (which will be explained in an extended version of the paper): We have a topological space $(\widehat{\mathcal{K}}^{\mathrm{fr}})^{-1}\widehat{\mathcal{L}}^{\mathrm{fr}}$ of fractional profinite framed links, which carries a natural map

$$h' : \mathcal{L}^{\mathrm{fr}} \to (\widehat{\mathcal{K}}^{\mathrm{fr}})^{-1}\widehat{\mathcal{L}}^{\mathrm{fr}}$$

and admits a continuous action of $\widehat{GT}$, hence of $G_{\mathbb{Q}}$ there. By DS and h', each 3-manifold M determines the subspace $L_M := h'(DS^{-1}(M))$ of $(\widehat{\mathcal{K}}^{\mathrm{fr}})^{-1}\widehat{\mathcal{L}}^{\mathrm{fr}}$. Examinations of the behavior of L_M's under our Galois action look significant to realize the analogies between number rings and 3-manifolds, which were suggested by Kapranov [Kap], Morishita [Mo] and Reznikov [R].

Extending our Galois action into an action on 'profinite 3-cobordisms' is another direction. The author expects that it would be a direct approach to realize Kapranov's [Kap] analogies between Langlands theory and Atiyah's TQFT (topological quantum field theory).

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